M.Sc. MATHEMATICS – I YEAR

DKM13 : DIFFERENTIAL EQUATIONS

SYLLABUS

Unit I : Second order linear equations - The general solution of a homogeneous equation - Use of a known solution to find another - The method of variation of parameters - Power series solution - Series solution of a first order equation.

Unit II : Second order linear equations - Ordinary points - regular singular points - Legendre polynomials.

Unit III : Bessel functions and Gamma functions - Linear systems - Homogeneous linear systems with constant coefficients - The method of successive approximation - Picard's theorem.

Unit IV : Partial differential Equations - Cauchy's problem for first order equations - Linear equations of first order - Nonlinear partial differential equations of first order - Cauchy's method of Characteristics - Compatible system of first order equations.

Unit V : Charpit's method - Special types of first order equations - Solutions satisfying given conditions - Jacobi's method - Linear partial differential equations with constant coefficients - Equation with variable coefficients.

Unit - I
Ordinary Differential Equations

Linear differential equations of Second order

The general second order linear differential equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

Where P(x), Q(x), R(x) are functions of x or constants.

For convenience we write the equation is

$y'' + P(x)\, y' + Q(x)\, y = R(x)$

The solution of the above equation has 2 parts namely one corresponding to R(x) = 0 and the other corresponding to R(x) as a function of x (or) constant.

The solution corresponding to R(x) = 0

ie) the solution of $y'' + P(x)\, y' + Q(x)\, y = 0$ is the general solution and it contains two arbitrary constant.

The solution corresponding to the particular function R(x) is called the particular integral of the equation.

The complete solution of the equation

$y'' + P(x)\, y' + Q(x)\, y = R(x)$ is $y = y_g + y_p$

Where y_g is the general solution of the equation $y'' + P(x)\, y' + Q(x)\, y = 0$ and y_p is the particular integral corresponding to R(x).

Consider the Second order linear differential equation

$y'' + P(x)\, y' + Q(x)\, y = R(x)$ …..(1)

Equation (1) is said to be non-homogeneous and

$y'' + P(x)\, y' + Q(x)\, y = 0$ …..(2)

Equation (2) is said to be homogeneous.

The general solution of equation (2) is taken as y_g and the particular solution of equation (1) is taken as y_p.

y_g contains two arbitrary constants as it is the solution of the 2nd order linear differential equation in equation (1).

Theorem:

If y_g is the general solution of $y'' + P(x)\, y' + Q(x)\, y = 0$ and y_p is any particular solution of the equation $y'' + P(x)\, y' + Q(x)\, y = R(x)$. Then $y_g + y_p$ is the general solution of $y'' + P(x)\, y' + Q(x)\, y = R(x)$.

Proof:

Let $y'' + P(x)\, y' + Q(x)\, y = 0$ …..(1)

be the homogeneous equation.

and $y'' + P(x)\, y' + Q(x)\, y = R(x)$ …..(2)

be the non-homogeneous equation.

Given y_g is the general solution of (1)

$\therefore\ y_g'' + P(x)\, y_g' + Q(x)\, y_g = 0$ …..(3)

also y_p is the particular solution of (2)

$\therefore\ y_p'' + P(x)\, y_p' + Q(x)\, y_p = R(x)$ …..(4)

(3) + (4)

$\rightarrow (y_g'' + y_p'') + P(x)\, [y_g' + y_p'] + Q(x)\, [y_g + y_p] = R(x)$

$\rightarrow (y_g + y_p)'' + P(x)\, (y_g + y_p)' + Q(x)\, (y_g + y_p) = R(x)$

This shows that $y_g + y_p$ is the general solution of (2).

Theorem:

If $y_1(x)$ and $y_2(x)$ are any two solutions of $y'' + P(x)\, y' + Q(x)\, y = 0$. Then $c_1\, y_1(x) + c_2\, y_2(x)$ is also a solution for any constants c_1 and c_2.

Proof:

Given $y_1(x)$ and $y_2(x)$ are the solution of $y'' + P(x)\, y' + Q(x)\, y = 0$ …..(1)

$\therefore\ y_1''(x) + P(x)\, y_1'(x) + Q(x)\, y_1(x) = 0$ …..(2)

$y_2''(x) + P(x)\, y_2'(x) + Q(x)\, y_2(x) = 0$ …..(3)

T.P. $c_1\, y_1(x) + c_2\, y_2(x)$ is the solution of (1) ie) t.p $c_1 y_1(x) + c_2 y_2(x)$ satisfies equation (1)

Now, $[c_1 y_1(x) + c_2 y_2(x)]'' + P(x) [c_1 y_1(x) + c_2 y_2(x)]' + Q(x) [c_1 y_1(x) + c_2 y_2(x)]$
$= c_1 y_1''(x) + c_2 y_2''(x) + P(x) c_1 y_1'(x) + P(x) c_2 y_2'(x) + Q(x) c_1 y_1(x) + Q(x) c_2 y_2(x)$

$= c_1 (y_1''(x) + P(x) y_1'(x) + Q(x) y_1(x)) + c_2 (y_2''(x) + P(x) y_2'(x) + Q(x) y_2(x))$

$= c_1 (0) + c_2 (0)$ (using (2) + (3))

$= 0.$

This shows that $c_1 y_1(x) + c_2 y_2(x)$ satisfies equation (1).

$\therefore$ $c_1 y_1(x) + c_2 y_2(x)$ is the solution of equation (1).

Problem:

By inspection find the general solution of $y'' = e^x$

Solution:

Given $y'' = e^x$

$y' = e^x + c_1$

$y = e^x + c_1 x + c_2$

Problem:

By eliminating the constants c_1& c_2 find the differential equation of each of the following families of curves.

1) $y = c_1 x + c_2 x^2$

2) $y = c_1 e^{kx} + c_2 e^{-kx}$

3) $y = c_1 \sin kx + c_2 \cos kx$

Solution:

1) $y = c_1 x + c_2 x^2$

$y' = c_1 + 2 c_2 x$

$y'' = 2c_2$

$y' = c_1 + y'' x$

$c_1 = y' - y''x$

$$\therefore y = (y' - y''x)\,x + \frac{y''}{2}x^2$$

$$y = y'x - y''x^2 + \frac{y''}{2}x^2$$

$$y = \frac{2y'x - 2y''x^2 + y''x^2}{2}$$

$2y = 2y'x - y''x^2$

$y''x^2 - 2y'x + 2y = 0.$

2) $y = c_1 e^{kx} + c_2 e^{-kx}$

$y' = c_1 e^{kx}.k + c_2 e^{-kx} (-k)$

$y'' = c_1 k e^{kx} (k) - c_2 k e^{-kx} (-k)$

$y'' = k^2 c_1 e^{kx} + k^2 c_2 e^{-kx}$

$y'' = k^2 (c_1 e^{kx} + c_2 e^{-kx})$

$y'' = k^2 y$

$y'' - k^2 y = 0.$

3) $y = c_1 \sin kx + c_2 \cos kx$

$y' = c_1 \cos kx\,(k) + c_2 (-\sin kx)\,k.$

$y' = c_1 k \cos kx - c_2 k \sin kx$

$y'' = c_1 k (-\sin kx)\,k - c_2 k \cos kx.k$

$y'' = -c_1 k^2 \sin kx - c_2 k^2 \cos kx$

$y'' = -k^2 (c_1 \sin kx + c_2 \cos kx)$

$y'' = -k^2 y$

$y'' + k^2 y = 0.$

Problem:

Verify that $y = c_1x^{-1} + c_2x^{+5}$ is a solution of $x^2 y'' - 3xy' - 5y = 0$ on any interval [a, b] that does not contain zero. If $x_0 \neq 0$ and if y_0 and y_0' are arbitrary. Show that c_1 and c_2 can be chosen in one and only one way. So that $y(x_0) = y_0$ and $y'(x_0) = y_0'$

Solution:

Given $x^2 y'' - 3xy' - 5y = 0$ .....(1)

Take $y_1 = x^{-1}$, $y_2 = x^5$

$$\therefore y_1 = \frac{1}{x}$$

When $x = 0$, We find y_1 is not continuous and so it is not differentiable.

$\therefore$ In any [a,b] which does not contain zero.

If $x_0 \neq 0$

y_1 is differentiable

Let $y_1 = x^{-1}$ $y_2 = x^5$

$$y_1 = \frac{1}{x}$$

$$y_1' = -\frac{1}{x^2}$$

$$y_1'' = \frac{2}{x^3}$$

$y_2 = x^5$

$y_2' = 5x^4$

$y_2'' = 20x^3$

T.P y_1 and y_2 are the solution of (1)

Now $x^2 y_1'' - 3x y_1' - 5y_1$

$$= x^2 \frac{2}{x^3} - 3x\left(\frac{-1}{x^2}\right) - 5\left(\frac{1}{x}\right)$$

$$= \frac{2}{x} + \frac{3}{x} - \frac{5}{x}$$

$x^2 y_1'' - 3x y_1' - 5y_1 = 0$

Also, $x^2 y_2'' - 3x y_2' - 5y_2$

$= x^2\, 20x^3 - 3x \,.\, 5x^4 - 5x^5$

$= 20x^5 - 15x^5 - 5x^5 \qquad = 0.$

$\therefore$ y_1 and y_2 are the solution of (1)

$\therefore$ $y = c_1x^{-1} + c_2x^5$ is the general solution of (1)

Given $y(x_0) = y_0$, $y'(x_0) = y_0$

We've $y = c_1x^{-1} + c_2x^5$

$y(x_0) = c_1x_0^{-1} + c_2x_0^5$

$y'(x_0) = -c_1x_0^{-2} + 5c_2x_0^4$

$y_0 = c_1x_0^{-1} + c_2x_0^5$

$y_0' = -c_1x_0^{-2} + 5c_2x_0^4$

T.P c_1 and c_2 are chosen in one and only one way

$$\begin{vmatrix} x_0^{-1} & x_0^{5} \\ -x_0^{-2} & 5x_0^{4} \end{vmatrix} = 5x_0^{3} + x_0^{3}$$

$$= 6x_0^3 \qquad \neq 0.$$

$\therefore$ c_1 and c_2 can be chosen in an one and only one way.

The general solution of the homogeneous equation:

If two functions $f(x)$ & $g(x)$ are defined on an interval [a, b] and have a property that one is the constant multiple of the other then they are said to be linearly dependent on [a, b].

Otherwise that is if neither is a constant multiple of the other they are called linearly independent.

If $f(x)$ is identically zero, then $f(x)$ and $g(x)$ are linearly dependent for every function $g(x)$, since $f(x) = o.\, g(x)$.

If y_1 and y_2 are the solutions on the [a,b]. Then the wronskian denoted by $W(y_1, y_2)$ and defined by $W(y_1, y_2) = y_1 y_2' - y_1' y_2$.

Theorem:

Let $y_1(x)$ and $y_2(x)$ be linearly independent solution of the homogeneous equation

$y'' + P(x)\, y' + Q(x)\, y = 0$ .....(1)

on the interval [a, b].

Then $c_1\, y_1(x) + c_2\, y_2(x)$ .....(2)

is the general solution of equation (1) on the [a, b]. In the sense that every solution of equation (1) on this interval can be obtained from equation (2) by a suitable choice of the arbitrary constant c_1 and c_2.

Proof:

The proof will be given in stages by means of several lemma's and auxiliary ideas.

Let $y(x)$ be any solution of equation (1) on the [a, b] we must show that the constant c_1 and c_2 can be found so that $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$ for all x in [a, b].

Lemma: 1

If $y_1(x)$ and $y_2(x)$ are any two solution of $y'' + P(x)\, y' + Q(x)\, y = 0$ on [a, b]. Then their Wronskian $W = W(y_1, y_2)$ is either identically zero or never zero on [a, b].

Let y_1 and y_2 be the two solutions of

$y'' + P(x)\, y' + Q(x)\, y = 0$ .....(1)

$\therefore\ y_1'' + P(x)\, y_1' + Q(x)\, y_1 = 0$ .....(3)

$y_2'' + P(x)\, y_2' + Q(x)\, y_2 = 0$ .....(4)

$(3) \times y_2 \rightarrow y_1''\, y_2 + P(x)\, y_1'\, y_2 + Q(x)\, y_1\, y_2 = 0$ .....(5)

$(4) \times y_1 \rightarrow y_1\, y_2'' + P(x)\, y_1\, y_2' + Q(x)\, y_1\, y_2 = 0$ .....(6)

$(6) - (5) \rightarrow (y_2''\, y_1 - y_1''\, y_2) + P(x)\, (y_1\, y_2' - y_1'\, y_2) = 0$ .....(7)

w.k.t.

$W = y_1\, y_2' - y_1'\, y_2$

$W^1 = y_1\, y_2'' + y_1'\, y_2' - y_1'\, y_2' - y_1''\, y_2$

$= y_1\, y_2'' - y_1''\, y_2$

$(7) \rightarrow W^1 + P(x)\, W = 0$

$W^1 = -P(x)\, W$

$$\frac{dW}{dx} = -P(x)W$$

$$\frac{dW}{W} = -P(x)\,dx$$

$\int ing$

$$\int \frac{dW}{W} = -\int P(x)\,dx$$

$$\rightarrow \log W = \log e^{-\int P(x)\,dx} + \log c$$

$$\Rightarrow \log W = \log ce^{-\int P(x)\,dx}$$

$$\Rightarrow W = ce^{-\int P(x)\,dx}$$

Since the exponential factor is never zero the proof is complete.

Lemma: 2

If $y_1(x)$ and $y_2(x)$ are two solutions of $y'' + P(x)\,y' + Q(x)\,y = 0$ on the [a, b] then they are linearly dependent on this interval iff the wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is identically zero

Assume that y_1 and y_2 are linearly dependent.

T.P $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0$

If either function is identically zero on [a, b]

Clearly the Wronskian is zero Now we assume w.l.g. that neither is identically zero.

Since y_1 and y_2 are linearly dependent each function in a constant multiple of the other

$\therefore$ We've $y_2 = c\, y_1$ for some constant c

$y_2' = c\, y_1'$

$\therefore W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$= y_1\, cy_1' - y_1'\, cy_1$

$= 0.$

$\therefore$ W is identically zero

Conversely,

Assume that the wronskian is identically zero

T.P y_1& y_2 are linearly dependent. If y_1 is identically zero on the [a, b] then the functions are linearly dependent.

$\therefore$ We may assume that ($y_1 \neq 0$) identically on the [a, b]

$\therefore$ y_1 does not vanish at all on some subinterval [c, d] of [a, b].

Since the wronskian is identically zero on the [a, b] we can divide it by y_1^2

We get, $\frac{W}{y_1^2} = 0$

$$\frac{y_1 y_2' - y_1' y_2}{y_1^2} = 0$$

$$\rightarrow d\left(\frac{y_2}{y_1}\right) = 0$$

$\int ing$

$$\int d\left(\frac{y_2}{y_1}\right) dx = 0$$

$$\rightarrow \frac{y_2}{y_1} = k$$

$$\rightarrow y_2 = k\, y_1$$

for some constant k and all x in [c, d].

$\therefore$ Since $y_2(x)$ and $k\, y_1(x)$ have equal values in [c, d]

$\therefore$ $y_2(x) = k\, y_1(x)$ for all x in [a, b]

$\therefore$ y_1 and y_2 are linearly dependent on the [a, b]

Hence the lemma

Since $c_1 y_1(x) + c_2 y_2(x)$ and $y(x)$ are both solutions of equation (1) on the [a, b]

It suffices to show that for some point x_0 in the [a, b] we can find c_1 and c_2 so that

$c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0)$ and $c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$

For this system to be solved for c_1 and c_2

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1^1(x_0) & y_2^1(x_0) \end{vmatrix} = y_1(x_0)\, y_2'(x_0) - y_1'(x_0)\, y_2(x_0) \neq 0.$$

The above result showing that the Wronskian of any two linearly independent solutions of equation (1) is not identically zero.

ie) y_1 and y_2 are linearly independent solutions of equation (1) iff $W \neq 0$.

Problem:

Show that e^x and e^{-x} are linearly independent solution of $y'' - y = 0$ on any interval.

Solution:

First T.P e^x and e^{-x} is a solution of $y'' - y = 0$

$y_1 = e^x$, $y_2 = e^{-x}$

$y_1' = e^x$ $y_2' = -e^{-x}$

$y_1'' = e^x$ $y_2'' = e^{-x}$

Now $y_1'' - y_1 = e^x - e^x$

$= 0.$

$\therefore$ $y_1 = e^x$ is the solution of $y'' - y = 0$

Also $y_2'' - y_2 = e^{-x} - e^{-x}$

$= 0$

$\therefore$ $y_2 = e^{-x}$ is the solution of $y'' - y = 0$

Next T.P y_1& y_2 are Linearly Independent solution

ie) T.P $W(y_1, y_2) \neq 0$

$W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$= -e^x e^{-x} - e^x e^{-x}$

$= -1 - 1$

$= -2$

$\neq 0.$

$\therefore$ y_1& y_2 are linearly independent solution.

Problem:

Show that $y = c_1 e^x + c_2 e^{2x}$ is the general solution of $y'' - 3y' + 2y = 0$ on any interval and find a particular solution for which $y(0) = -1$ and $y'(0) = 1$.

Solution:

Given: $y'' - 3y' + 2y = 0$ …..(1)

T.P y_1 and y_2 are solution of $y'' - 3y' + 2y = 0$

$y_1 = e^x$ $y_2 = e^{2x}$

$y_1' = e^x$ $y_2' = 2e^{2x}$

$y_1'' = e^x$ $y_2'' = 4e^{2x}$

Now $y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x$

$= 3e^x - 3e^x$

$= 0$

$\therefore$ y_1 is the solution of $y'' - 3y' + 2y = 0$

Also $y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 3 \times 2e^{2x} + 2e^{2x}$

$= 4e^{2x} - 6e^{2x} + 2e^{2x}$

$= 6e^{2x} - 6e^{2x}$

$= 0.$

$\therefore$ y_2 is the solution of $y'' - 3y' + 2y = 0$

$\therefore$ y_1& y_2 are the solution of (1)

Now, $\frac{y_2}{y_1} = \frac{e^{2x}}{e^x} = e^x$ is not a constant

$\therefore$ y_1 or y_2 cannot be written as one is the constant multiple of the other.

$\therefore$ y_1 and y_2 are Linearly independent

Also $W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$= e^x 2e^{2x} - e^x e^{2x}$

$= 2e^{3x} - e^{3x}$

$= e^{3x} \neq 0.$

$\therefore$ y_1 & y_2 are linearly independent

$\therefore$ The general solution is $y(x) = c_1 e^{x} + c_2 e^{2x}$

Given $y(0) = -1$ and $y'(0) = 1$

We've $y(x) = c_1 e^{x} + c_2 e^{2x}$

$y'(x) = c_1 e^{x} + 2c_2 e^{2x}$

$y(0) = c_1 e^{o} + c_2 e^{o}$

$y'(0) = c_1 e^{o} + 2c_2 e^{o}$

$c_1 + c_2 = -1$ …..(2)

$c_1 + 2c_2 = 1$ …..(3)

Solving (2) & (3)

$c_1 + c_2 = -1$

$c_1 + 2c_2 = 1$

$-c_2 = -2$

$\boxed{c_2 = 2}$

$c_1 + c_2 = -1$

$c_1 = -1 - c_2$

$c_1 = -1 - 2$

$c_1 = -3$

$\therefore$ The particular solution is $y = -3e^{x} + 2e^{2x}$.

Problem:

Consider the two functions $f(x)$ x^3 and $g(x) = x^2|x|$ on the closed interval (-1, 1)

a) Show that their Wronskian W (f,g) vanishes identically.

b) Show that f and g are not linearly dependent.

c) Do Part (a) & (b) contradictors lemma 2 if not, why not

Solution:

a) On the interval $-1 \leq x < 0$

$f(x) = x^3$, $g(x) = x^2(-x)$

ie) $f(x) = x^3$, $g(x) = -x^3$

$f'(x)\ 3x^2$ $g'(x) = -3x^2$

$W(f,g) = fg' - f'g$

$= x^3(-3x^2) - 3x^2(-x^3)$

$= -3x^5 + 3x^5$

$= 0.$

$\therefore W(f,g) = 0$

At $x = 0$

Clearly $W(f,g) = 0$

On $0 < x \leq 1$

$f(x) = x^3$, $g(x) = x^2(x) = x^3$

$f'(x) = 3x^2$ $g'(x) = 3x^2$

$W(f,g) = fg' - f'g$

$= x^3(3x^2) - (3x^2)\,x^3$

$= 0.$

$\therefore W(f,g) = 0$ on $[-1,1]$

b) $\dfrac{f(x)}{g(x)} = \dfrac{x^3}{x^2|x|}$

$= \dfrac{x}{|x|}$

$= \pm 1$

Which is not a unique constant $\therefore$ f(x) and g(x) are not linearly dependent.

c) Part (a) & Part (b) are not a contradiction to lemma 2 for the following reasons.

g(x) = x^2 $|x|$ cannot be differentiable and f(x), g(x) cannot be the solutions of the homogeneous equation.

Problem: 6

It is clear that, sin$x$, cos$x$ and sin$x$, sin$x$ – cos$x$ are two distinct pairs of linearly independent solutions of y'' + y = 0. Thus if y_1 and y_2 are linearly independent solution of the homogeneous equation y'' + P(x) y' + Q(x) y = 0 we see that y_1 and y_2 are not uniquely determine by the equation.

a) Show that $\frac{-(y_1 y_2''-y_2 y_1'')}{W(y_1,y_2)}$ and $Q(x)\frac{y_1' y_2''-y_2' y_1''}{W(y_1,y_2)}$ so that the equation is uniquely determine by any given pair of linearly independent solutions.

b) Use part (a) to reconstruct the equation y'' + y = 0 from each of the two pairs of linearly independent solutions mentioned above.

c) Use part (a) to reconstruct the equation y'' – 4y' + 4y = 0 from the pair of linearly independent solutions e^{2x}, xe^{2x}.

Solution:

y_1 = sin$x$ $\quad\quad$ y_2 = cos$x$

y_1' = cos$x$ $\quad\quad$ y_2' = - sin$x$

y_1'' = - sin$x$ $\quad\quad$ y_2'' = - cos$x$

Now,

$y_1'' + y_1$ = - sin$x$ + sin$x$

= 0

$\therefore$ y_1 = sin$x$ is the solution of y'' + y = 0

$y_2'' + y_2$ = - cos$x$ + cos$x$

= 0

$\therefore$ y_2 = cos$x$ is the solution of y'' + y = 0

$y_3 = \sin x$ $\quad\quad$ $y_4 = \sin x - \cos x$

$y_3' = \cos x$ $\quad\quad$ $y_4' = \cos x + \sin x$

$y_3'' = -\sin x$ $\quad\quad$ $y_4'' = -\sin x + \cos x$

$y_3'' + y_3 = -\sin x + \sin x$

$= 0$

$\therefore$ $y_3 = \sin x$ is the solution of $y'' + y = 0$

$y_4'' + y_4 = -\sin x + \cos x + \sin x - \cos x$

$= 0$

$\therefore$ $y_4 = \sin x - \cos x$ is the solution of $y'' + y = 0$

$W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$= \sin x(-\sin x) - \cos x(\cos x)$

$= -\sin^2 x - \cos^2 x$

$= -(\sin^2 x + \cos^2 x)$

$= -1$

$\neq 0.$

$W(y_3, y_4) = y_3 y_4' - y_3' y_4$

$= \sin x(\cos x + \sin x) - \cos x(\sin x - \cos x)$

$= \sin x \cos x + \sin^2 x - \sin x \cos x + \cos^2 x$

$= \sin^2 x + \cos^2 x$

$= 1$

$\neq 0.$

$\therefore$ y_1, y_2, y_3 & y_4 are Linearly independent

a) Let y_1 & y_2 be the solutions of

$y'' + P(x) y' + Q(x) y = 0$

$\therefore y_1'' + P(x)\, y_1' + Q(x)\, y_1 = 0$ …..(1)

$y_2'' + P(x)\, y_2' + Q(x)\, y_2 = 0$ …..(2)

$(1) \times y_2 \rightarrow y_2\, y_1'' + P(x)\, y_1'\, y_2 + Q(x)\, y_1\, y_2 = 0$ …..(3)

$(2) \times y_1 \rightarrow y_2''\, y_1 + P(x)\, y_2'\, y_1 + Q(x)\, y_1\, y_2 = 0$ …..(4)

(4) – (3)

$\rightarrow y_1\, y_2'' - y_1''\, y_2 + P(x)\, (y_1\, y_2' - y_1'\, y_2) = 0$

$\rightarrow y_1\, y_2'' - y_1''\, y_2 + P(x)\, W(y_1, y_2) = 0$

$P(x)\, W(y_1, y_2) = -(y_1\, y_2'' - y_1''\, y_2)$

$$P(x) = \frac{-(y_1 y_2'' - y_1'' y_2)}{W(y_1, y_2)}$$

$(1) \rightarrow y_1'' + P(x)\, y_1' + Q(x)\, y_1 = 0$

$Q(x)\, y_1 = -y_1'' - P(x)\, y_1'$

$$= -y_1'' + \frac{(y_1 y_2'' - y_1'' y_2)\, y_1'}{y_1 y_2' - y_1' y_2}$$

$$= \frac{-y_1''(y_1 y_2' - y_1' y_2) + (y_1 y_2'' - y_1'' y_2)\, y_1'}{W(y_1, y_2)}$$

$$Q(x)\, y_1 = \frac{-y_1 y_1'' y_2' + y_1' y_1'' y_2 + y_1 y_1' y_2'' - y_1' y_1'' y_2}{W(y_1, y_2)}$$

$$Q(x)\, y_1 = \frac{y_1 y_1' y_2'' - y_1 y_1'' y_2'}{W(y_1, y_2)}$$

$$Q(x)\, y_1 = \frac{y_1 (y_1' y_2'' - y_1'' y_2')}{W(y_1, y_2)}$$

$$Q(x) = \frac{y_1' y_2'' - y_1'' y_2'}{W(y_1, y_2)}$$

Since y_1 & y_2 are Linearly Independent

$W(y_1, y_2) \neq 0.$

b) $y_1 = \sin x$ $y_2 = \cos x$

$y_1' = \cos x$ $y_2' = -\sin x$

$y_1'' = -\sin x$ $y_2'' = -\cos x$

$$P(x) = \frac{-(y_1 y_2'' - y_1'' y_2)}{W(y_1, y_2)}$$

$$= \frac{-[\sin x(-\cos x) - (-\sin x)(\cos x)]}{-1}$$

$$= \frac{-[-\sin x \cos x + \sin x \cos x)]}{-1}$$

$= 0.$

$P(x) = 0.$

$$Q(x) = \frac{y_1' y_2'' - y_1'' y_2'}{W(y_1, y_2)}$$

$$= \frac{-\cos x \cos x - (-\sin x)(-\sin x)}{-1}$$

$$= \frac{-\cos^2 x - \sin^2 x}{-1}$$

$$= \frac{-(\cos^2 x + \sin^2 x)}{-1}$$

$Q(x) = 1.$

$\therefore y'' + P(x)\, y' + Q(x)\, y = 0$

$y'' + 0.y' + 1.y = 0$

$y'' + y = 0.$

$y_1 = \sin x$ $y_2 = \sin x - \cos x$

$y_1' = \cos x$ $y_2' = \cos x + \sin x$

$y_1'' = -\sin x \qquad y_2'' = -\sin x + \cos x$

$$P(x) = \frac{-(y_1 y_2'' - y_1'' y_2)}{W(y_1, y_2)}$$

$$P(x) = \frac{-[\sin x(-\sin x + \cos x) - (-\sin x)(\sin x - \cos x)]}{1}$$

$= -[-\sin^2 x + \sin x \cos x - (-\sin^2 x + \sin x \cos x)]$

$= -[\sin^2 x + \sin x \cos x + \sin^2 x - \sin x \cos x]$

$= 0.$

$$Q(x) = \frac{y_1' y_2'' - y_1'' y_2'}{W(y_1, y_2)}$$

$$= \frac{\cos x(-\sin x + \cos x) + \sin x(\cos x + \sin x)}{1}$$

$= -\sin x \cos x + \cos^2 x + \sin x \cos x + \sin^2 x$

$= \cos^2 x + \sin^2 x$

$= 1.$

$\therefore\ y'' + P(x)\, y' + Q(x)\, y = 0$

$y'' + 0.y' + 1.y = 0$

$y'' + y = 0.$

c) $y_1 = e^{2x} \qquad y_2 = xe^{2x}$

$y_1' = 2e^{2x} \qquad y_2' = 2xe^{2x} + e^{2x}$

$y_1'' = 4e^{2x} \qquad y_2'' = 2[2xe^{2x} + e^{2x}] + 2e^{2x}$

$y_2'' = 4xe^{2x} + 4e^{2x}$

$W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$= e^{2x}(2xe^{2x} + e^{2x}) - 2e^{2x} . xe^{2x}$

$= 2xe^{4x} + e^{4x} - 2xe^{4x}$

$= e^{4x}$.

$$P(x) = \frac{-(y_1 y_2'' - y_1'' y_2)}{W(y_1, y_2)}$$

$$= \frac{-[e^{2x}(4xe^{2x} + 4e^{2x}) - 4e^{2x} xe^{2x}]}{e^{4x}}$$

$$= \frac{-[4xe^{4x} + 4e^{4x} - 4xe^{4x}]}{e^{4x}}$$

$$P(x) = \frac{-4e^{4x}}{e^{4x}}$$

$P(x) = -4$

$$Q(x) = \frac{y_1' y_2'' - y_1'' y_2'}{W(y_1, y_2)}$$

$$= \frac{2e^{2x}(4xe^{2x} + 4e^{2x}) - 4e^{2x}(2xe^{2x} + e^{2x})}{e^{4x}}$$

$$= \frac{8xe^{4x} + 8e^{4x} - 8xe^{4x} - 4^{p} e^{4x}}{e^{4x}}$$

$$= \frac{4e^{4x}}{e^{4x}}$$

$= 4$.

$\therefore$ y'' + P(x) y' + Q(x) y = 0

y'' + (-4) y' + 4y = 0

y'' - 4y' + 4y = 0.

The use of a known solution to find another:

Suppose y_1 is the known solution of the homogeneous equation

$$y'' + P(x)\, y' + Q(x)\, y = 0 \quad \text{.....(1)}$$

We've to find the other solution y_2 s.t y_1 and y_2 are linearly independent

The general solution is $y = c_1 y_1 + c_2 y_2$

Let us assume that y_2 vy_1 be the required solution.

$$\therefore\ y_2'' + P(x)\, y_2' + Q(x)\, y_2 = 0 \quad \text{.....(2)}$$

We've $y_2 = vy_1$

$y_2' = vy_1' + v'y_1$

$y_2'' = vy_1'' + v'y_1' + v'y_1' + v''y_1$

$y_2'' = vy_1'' + 2v'y_1' + v''y_1$

Sub in equation (2)

$$vy_1'' + 2v'y_1' + v''y_1 + P(x)\,[vy_1' + v'y_1] + Q(x)\, vy_1 = 0$$

$$v''y_1 + v'\,[2y_1 + P(x)\, y_1] + v[y_1'' + P(x)\, y_1' + Q(x)\, y_1] = 0$$

Divide by $v'y_1$

$$\to \frac{v''}{v'} + \frac{2y_1'}{y_1} + P(x) = 0$$

$$\to \frac{v''}{v'} + \frac{2y_1'}{y_1} = -P(x)$$

$\int ing$

$$\log v' + 2\log y_1 = -\int P(x)\, dx$$

$$\log v' = -2\log y_1 - \int P(x)\, dx$$

$$= -\log y_1^{\,2} - \int P(x) dx$$

$$= \log \frac{1}{y_1^{\,2}} + \log e^{-\int P(x)dx}$$

$$\log v' = \log \frac{1}{y_1^2} e^{-\int P(x)dx}$$

$$v' = \frac{1}{y_1^2} e^{-\int P(x)dx}$$

$\int ing$

$$v = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx.$$

Let us prove that y_1& y_2 are Linearly independent

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$= y_1 (vy_1' + v'y_1) - y_1' vy_1$$

$$= vy_1 y_1' + v'y_1^2 - y_1' vy_1$$

$$= v' y_1^2$$

$$= \frac{1}{y_1^2} e^{-\int P(x)dx} . y_1^2$$

$$= e^{-\int P(x)dx} \qquad \neq 0.$$

$\therefore$ y_1& y_2 are linearly independent.

Problem:

Verify that $y_1 = x^2$ is one solution of $x^2 y'' + xy' - 4y = 0$ and find the general solution.

Solution:

Given: $x^2y'' + xy' - 4y = 0$ …..(1)

$$\rightarrow y'' + \frac{1}{x} y' - \frac{4}{x^2} y = 0$$

T.P $y_1 = x^2$ is the solution of (1)

$y_1 = x^2$ $y_1' = 2x$ $y_1'' = 2.$

$x^2y_1'' + xy_1' - 4y_1 = x^2.2 + x.2x - 4x^2$

$= 4x^2 - 4x^2$

$= 0.$

$\therefore$ $y_1 = x^2$ is the solution of (1).

To find y_2

ie) $y_2 = vy_1$

Where $v = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx$

$$P(x) = \frac{1}{x}$$

$$v = \int \frac{1}{(x^2)^2} e^{-\int \frac{1}{x}dx} dx$$

$$= \int \frac{1}{x^4} e^{-\log x} dx$$

$$= \int \frac{1}{x^4} e^{\log\frac{1}{x}} dx$$

$$= \int \frac{1}{x^4}.\frac{1}{x}.dx$$

$$= \int x^{-5} dx$$

$$= \frac{x^{-5+1}}{-5+1}$$

$$= \frac{x^{-4}}{-4}$$

$$= \frac{1}{-4x^4}$$

$$\therefore v = -\frac{1}{4x^4}$$

$y_2 = vy_1$

$$= -\frac{1}{4x^4}.x^2\left(-\frac{1}{4x^2}\right)$$

$$y_2 = \frac{-1}{4x^2}$$

The general solution is

$y = c_1 y_1 + c_2 y_2$

$$y = c_1 x^2 + c_2\left(-\frac{1}{4x^2}\right)$$

1. $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$. Find the general solution.

Solution:

Given $x^2 y'' + xy' - y = 0$ …..(1)

$$\Rightarrow y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$$

Since $P(x) = \frac{1}{x}$,

To find y_2

ie) $y_2 = vy_1$

Where $v = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx$

$$= \int \frac{1}{x^2} e^{-\int\left(\frac{1}{x}\right)dx} dx$$

$$= \int \frac{1}{x^2} e^{-\log x} dx$$

$$= \int \frac{1}{x^2} e^{\log\frac{1}{x}} dx$$

$$= \int \frac{1}{x^2} . \frac{1}{x} dx$$

$$= \int x^{-3} dx$$

$$= \frac{x^{-3+1}}{-3+1}$$

$$= \frac{x^{-2}}{-2}$$

$$v = -\frac{1}{2x^2}$$

$y_2 = vy_1$

$$= -\frac{1}{2x^2} . x$$

$$y_2 = -\frac{1}{2x}$$

The general solution is

$y = c_1 y_1 + c_2 y_2$

$$= c_1 x + c_2\left(-\frac{1}{2x}\right)$$

$$y = c_1 x - \frac{1}{2} c_2 x^{-1}$$

2. Find y_2 and the general solution of each of the following equations from the given solution y_1

a) $y'' + y = 0$, $y_1 = \sin x$; b) $y'' - y = 0$, $y_1 = e^x$

Solution:

a) Given: $y'' + y = 0$

Since $P(x) = 0$

To find y_2

ie) $y_2 = vy_1$

Where $v = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx$

$$= \int \frac{1}{\sin^2 x} e^{-\int o dx} dx$$

$$v = \int \frac{1}{\sin^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx$$

$$= -\cot x$$

$y_2 = vy_1$

$$= -\cot x \times \sin x$$

$$= \frac{-\cos x}{\sin x} \times \sin x$$

$$y_2 = -\cos x$$

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \sin x + c_2 (-\cos x)$$

$$y = c_1 \sin x - c_2 \cos x$$

b) y'' – y = 0, $y_1 = e^x$

Given: y'' – y = 0

Since P(x) = 0

To find y_2

ie) $y_2 = vy_1$

Where $v = \int \frac{1}{y_1^2} e^{-\int P(x)} dx$

$$= \int \frac{1}{(e^x)^2} e^{-\int o} dx$$

$$= \int \frac{1}{e^{2x}} dx$$

$$= \int e^{-2x} dx$$

$$= \frac{-e^{-2x}}{2}$$

$y_2 = vy_1$

$$= \frac{-e^{-2x}}{2}.e^x$$

$$= \frac{-e^{-x}}{2}$$

The general solution is

$y = c_1 y_1 + c_2 y_2$

$$= c_1 e^x + c_2 \left(\frac{-e^{-x}}{2} \right)$$

$y = c_1 e^{x} - \frac{1}{2} c_2 e^{-x}$

Problem:

Verify that $y_1 = x$ is the solution of $y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$ Find the general

Solution:

Given: $y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$ …..(1)

T.P $y_1 = x$ is the solution of equation (1)

$y_1 = x$, $y_1' = 1$, $y_1'' = 0$

Now, $y_1'' - \frac{x}{x-1} y_1' + \frac{1}{x-1} y_1 = 0 - \frac{x}{x-1}(1) + \frac{x}{x-1}$

$$= -\frac{x}{x-1} + \frac{x}{x-1}$$

$$= 0.$$

$\therefore$ $y_1 = x$ is the solution of equation (1)

To find y_2

$y_2 = v y_1$

Where $v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\, dx$

$P(x) = -\frac{x}{x-1}$

$\therefore v = \int \frac{1}{x^2} e^{-\int (-\frac{x}{x-1})\,dx}\, dx$

$v = \int \frac{1}{x^2} e^{\int \frac{x}{x-1}\,dx}\, dx$

$$= \int \frac{1}{x^2} e^{\int\left(\frac{x-1+1}{x-1}\right)dx} dx$$

$$= \int \frac{1}{x^2} e^{\int\left(1+\frac{1}{x-1}\right)dx} dx$$

$$= \int \frac{1}{x^2} e^{x+\log(x-1)} dx$$

$$= \int \frac{1}{x^2} e^{x}.e^{\log(x-1)}\, dx$$

$$= \int \frac{1}{x^2} e^{x}(x-1)\, dx$$

$$= \int \frac{1}{x^2} xe^{x} dx - \int \frac{1}{x^2} e^{x}\, dx$$

$$= \int \frac{1}{x} e^{x} dx - \int \frac{1}{x^2} e^{x}\, dx$$

$$= \int x^{-1} e^{x}\, dx - \int x^{-2} e^{x}\, dx$$

$$= x^{-1} e^{x} + \int e^{x} x^{-2}\, dx - \int e^{x} x^{-2}\, dx$$

$u = x^{-1}$, $du = -x^{-2}\, dx$, $v = e^x$, $\int dv = \int e^x dx$

$$v = \frac{e^x}{x}$$

$\therefore\ y_2 = vy_1$

$$= \frac{e^x}{x} \times x$$

$\therefore\ y_2 = e^x$

$\therefore$ The general solution is

$y = c_1 y_1 + c_2 y_2,$

$y = c_1 x + c_2 e^x$

Problem:

Verify that $y_1 = x$ is the solution of the equation $(1 - x^2)\, y'' - 2xy' + 2y = 0$. Find the general solution

Solution:

Given: $(1 - x^2)\, y'' - 2xy' + 2y = 0$ …..(1)

$$\Rightarrow y'' - \frac{2x}{(1-x^2)} y' + \frac{2}{(1-x^2)} y = 0$$

T.P $y_1 = x$ is the solution of equation (1)

$y_1 = x$, $y_1'' = 1$, $y_1'' = 0$

$(1 - x^2)\, y_1'' - 2xy_1' + 2y_1$

$= (1 - x^2)(0) - 2x(1) + 2x$

$= -2x + 2x$

$= 0.$

$\therefore$ $y_1 = x$ is the solution of $y_1 = x$

To find y_2

$y_2 = vy_1$

Where $v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\, dx$

$$P(x) = -\frac{2x}{(1-x^2)}$$

$$v = \int \frac{1}{x^2} e^{-\int\left(\frac{-2x}{1-x^2}\right)dx}\, dx$$

$$= \int \frac{1}{x^2} e^{\int\left(\frac{2x}{1-x^2}\right)dx}\, dx$$

$$= \int \frac{1}{x^2} e^{-\log(1-x^2)}\, dx$$

$$= \int \frac{1}{x^2} e^{\log \frac{1}{1-x^2}} \, dx$$

$$= \int \frac{1}{x^2} \cdot \frac{1}{1-x^2} . dx$$

$$* \quad \frac{1}{x^2(1-x^2)} = \frac{A}{x^2} + \frac{B}{1-x^2}$$

$$= \int \left(\frac{1}{x^2} + \frac{1}{1-x^2} \right) dx$$

$1 = A(1 - x^2) + Bx^2$

$$= \int x^{-2} \, dx + \int \frac{1}{1-x^2} dx$$

Put $x = 1 \rightarrow B = 1$

$$= \frac{x^{-2+1}}{-2+1} + \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

$$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2} + \frac{1}{1-x^2}$$

$$v = -\frac{1}{x} + \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

$$\int \frac{1}{a^2 - x^2} dx$$

$$= \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right)$$

$\therefore$ $y_2 = vy_1$

$$= \left[-\frac{1}{x} + \frac{1}{2} \log \left(\frac{1+x}{1-x} \right) \right] x$$

$$= -1 + \frac{x}{2} \log \left(\frac{1+x}{1-x} \right)$$

$\therefore$ The general solution is

$y = c_1 y_1 + c_2 y_2$

$$y = cx + c_2 \left[-1 + \frac{x}{2} \log \left(\frac{1+x}{1-x} \right) \right]$$

The Method of Variation of Parameters.

To solve the Second order linear equations

$y'' + P(x) y' Q(x) y = R(x)$ …..(1)

The solution corresponding to $R(x) \neq 0$ is called a Particular solution

For this we consider the homogeneous equation

$y'' + P(x)\, y' + Q(x)\, y = 0$ …..(2)

The general solution of equation (2) is

$y = c_1 y_1 + c_2 y_2$

Where c_1& c_2 are arbitrary constant

The solution of equation (1) may be assume in the above form, where c_1& c_2 are taken as the unknown function v_1& v_2.

$\therefore$ The Particular solution of equation (1) is $y = v_1 y_1 + v_2 y_2$

The method applied is known as the variation of parameters.

We've $y = v_1 y_1 + v_2 y_2$ …..(3)

$y' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2$

$= (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2)$

Let us assume v_1 and v_2 be such that

$v_1' y_1 + v_2' y_2 = 0$ …..(a)

$y' = v_1 y_1' + v_2 y_2'$ …..(4)

$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'$ …..(5)

Sub (3), (4), (5) in (1)

$v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' + P(x)\,[v_1 y_1' + v_2 y_2'] + Q(x)\,[v_1 y_1 + v_2 y_2] = R(x)$

$v_1 [y_1'' + P(x)\, y_1' + Q(x)\, y_1] + v_2 [y_2'' + P(x)\, y_2' + Q(x)\, y_2] +$
$v_1' y_1' + v_2' y_2' = R(x)$ …..(6)

Since y_1& y_2 are solution of (2)

$\therefore\ y_1'' + P(x)\, y_1' + Q(x)\, y_1 = 0$

$y_2'' + P(x)\, y_2' + Q(x)\, y_2 = 0$

$\therefore$ Equation (6) becomes

$v_1 (0) + v_2 (0) + v_1' y_1' + v_2' y_2' = R(x)$

$v_1' y_1' + v_2' y_2' = R(x)$ …..(b)

Solving equation (a) & (b)

$v_1' y_1 + v_2' y_2 = 0$ …..(a)

$v_1' y_1' + v_2' y_2' = R(x)$ …..(b)

(a) × y_2' $\Rightarrow$ $v_1' y_1 y_2' + v_2' y_2 y_2' = 0$ …..(7)

(b) × y_2 $\Rightarrow$ $v_1' y_1' y_2 + v_2' y_2 y_2' = R(x) y_2$ …..(8)

(7) - (8) $\Rightarrow$ $v_1' [y_1 y_2' - y_1' y_2] = - R(x) y_2$

$v_1' W(y_1, y_2) = - R(x) y_2$

$$\therefore v_1^{1} = -\frac{R(x) y_2}{W(y_1, y_2)}$$

$\int ing$

$$v_1 = -\int \frac{R(x) y_2}{W(y_1, y_2)} dx$$

(a) → $v_2' y_2 = - v_1' y_1$

$$v_2' y_2 = \frac{R(x) y_2 y_1}{W(y_1, y_2)}$$

$$v_2' = \frac{R(x) y_1}{W(y_1, y_2)}$$

$\int ing$

$$v_2 = \int \frac{R(x) y_1}{W(y_1, y_2)} dx$$

Since y_1 & y_2 are Linearly independent solutions of the homogeneous equation (2).

$\therefore$ $W(y_1, y_2) \neq 0$.

$\therefore$ The expressions v_1' and v_2' are valid expressions.

$$\therefore v_1 = -\int \frac{R(x) y_2}{W(y_1, y_2)} dx \quad \text{and} \quad v_2 = \int \frac{R(x) y_1}{W(y_1, y_2)} dx$$

$\therefore$ The Particular solution of equation (1) is

$y = v_1 y_1 + v_2 y_2$

Note:

The complete solution in $y = c_1 y_1 + c_2 y_2 + y_p$ Where $y_p = v_1 y_1 + v_2 y_2$

Problem:

Find the particular solution of y'' - 2y' + y = 2x. First by inspection and then by variation of parameters.

Solution:

Given: $y'' - 2y' + y = 2x$ …..(1)

The homogeneous equation is $y'' - 2y' + y = 0$ …..(2)

The auxillary equation is

$m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$

$m = 1,1$

$\therefore$ The general solution is

$y = (c_1 + c_2x)\, e^x$

ie) $y = c_1e^x + c_2xe^x$

$\therefore\ y_1 = e^x \quad , \quad y_2 = xe^x$

$y_1' = e^x , \quad y_2' = xe^x + e^x$

$W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$= e^x (xe^x + e^x) - e^x xe^x$

$= xe^{2x} + e^{2x} - xe^{2x}$

$= e^{2x}$

$\neq 0.$

To find the particular solution of (1)

The Particular solution is

$y_p = v_1 y_1 + v_2 y_2$

$R(x) = 2x$

$$v_1 = -\int \frac{R(x)\, y_2}{W(y_1, y_2)} dx$$

$$= -\int \frac{2x(xe^x)}{e^{2x}} dx$$

$$= -2\int x^2 e^x e^{-2x} dx$$

$$= -2\int x^2 e^{-x} dx$$

$$= -2\left\{-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right\}$$

$u = x^2$ $\quad \int dv = \int e^{-x} dx$

$v_1 = 2e^{-x} (x^2 + 2x + 2)$

$u' = 2x \qquad v = -e^{-x}$

$$\left[\int udv = uv - u'v_1 + u''v_2 - u''v_2 + ... + (-1)^n u^n vn\right]$$

$u'' = 2 \qquad v_1 = e^{-x}$

$$v_2 = \int \frac{R(x)\, y_1}{W(y_1, y_2)} dx$$

$v_2 = -e^{-x}$

$$= \int \frac{2xe^x}{e^{2x}} dx$$

$$= 2\int xe^x e^{-2x} dx$$

$u = x$

$du = dx$

$$= 2\int xe^{-x} dx$$

$\int dv = \int e^{-x} dx$

$v = -e^{-x}$

$$= 2\left\{-xe^{-x} + \int e^{-x} dx\right\}$$

$= 2 (-xe^{-x} - e^{-x})$

$= -2e^{-x} (x + 1).$

$\therefore$ The Particular solution is

$y_p = v_1 y_1 + v_2 y_2$

$y_p = 2e^{-x} (x^2 + 2x + 2) e^x - 2e^{-x} (x + 1) xe^x$

$= 2 (x^2 + 2x + 2) - 2x (x + 1)$

$= 2x^2 + 4x + 4 - 2x^2 - 2x$

$y_p = 2x + 4.$

The complete solution is

$y = y_g + y_p$

$y = c_1 e^x + c_2 x e^x + 2x + 4.$

Problem

Find the Particular solution of $y'' + 4y = \tan 2x$

Solution:

Given: $y'' + 4y = \tan 2x$ …..(1)

The homogeneous equation is $y'' + 4y = 0$ …..(2)

The auxiliary equation is

$m^2 + 4 = 0$

$m^2 = -4$

$m^2 = i^2 2^2$

$m = \pm 2i$

The general solution is

$y = c_1 \cos 2x + c_2 \sin 2x$

$y_1 = \cos 2x \qquad y_2 = \sin 2x$

$y_1' = -2 \sin 2x \quad y_2' = 2 \cos 2x$

$W(y_1, y_2) = y_1 y_2' - y_1' y_2$

$= \cos 2x (2\cos 2x) + 2\sin 2x (\sin 2x)$

$= 2 \cos^2 2x + 2 \sin^2 2x$

$= 2 (\cos^2 2x + \sin^2 2x)$

$= 2. \quad \neq 0.$

To find the particular solution of equation (1)

The particular solution is

$y_p = v_1 y_1 + v_2 y_2$

$R(x) = \tan 2x$

$$v_1 = -\int \frac{R(x)\, y_2}{W(y_1, y_2)} dx$$

$$= -\int \frac{\tan 2x \sin 2x}{2} dx$$

$$= -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x}.\sin 2x\, dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \left(\frac{1}{\cos 2x} - \cos 2x\right) dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x)\, dx$$

$$= -\frac{1}{2}\left\{\log \frac{(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right\}$$

$$v_1 = -\log \frac{(\sec 2x + \tan 2x)}{4} + \frac{\sin 2x}{4}$$

$$v_2 = \int \frac{R(x)\, y_1}{W(y_1, y_2)} dx$$

$$v_2 = \int \frac{\tan 2x \cos 2x}{2} dx$$

$$= \frac{1}{2}\int \frac{\sin 2x}{\cos 2x}.\cos 2x\, dx$$

$$= \frac{1}{2}\int \sin 2x\ dx$$

$$= \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]$$

$$= -\frac{\cos 2x}{4}$$

$\therefore$ The Particular solution is

$y_p = v_1 y_1 + v_2 y_2$

$$y_p = \left[\frac{-\log(\sec 2x + \tan 2x)}{4} + \frac{\sin 2x}{4}\right]\cos 2x - \frac{\cos 2x}{4}\sin 2x$$

$$= -\log\frac{(\sec 2x + \tan 2x)}{4}\cos 2x + \frac{\sin 2x \cos 2x}{4} - \frac{\cos 2x \sin 2x}{4}$$

$$y_p = -\log\frac{(\sec 2x + \tan 2x)}{4}\cos 2x$$

Problem:

Find the general solution of $(x^2 + x)\, y'' + (2 - x^2)\, y' - (2 + x)\, y = x\,(x + 1)^2$

Solution:

Given: $(x^2 + x)\, y'' + (2 - x^2)\, y' - (2 + x)\, y = x\,(x + 1)^2$ …..(1)

The homogeneous equation is $(x^2 + x)\, y'' + (2 - x^2)\, y' - (2 + x)\, y = 0$ …..(2)

Take $y_1 = e^x$

T.P $y_1 = e^x$ is the solution of equation (2)

$y_1' = e^x$ $y_1'' = e^x$

$\therefore (x^2 + x)\, y_1'' + (2 - x^2)\, y_1' - (2 + x)\, y_1 = (x^2 + x)\, e^x + (2 - x^2)\, e^x - (2 + x)\, e^x$

$= x^2 e^x + xe^x + 2e^x - x^2 e^x - 2e^x - xe^x$

$= 0.$

$\therefore$ $y_1 = e^x$ is the solution of equation (2)

To find y_2

$y_2 = vy_1$

Where $v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx$

$$(1) \rightarrow y'' + \left(\frac{2-x^2}{x^2-x}\right)y' - \left(\frac{2+x}{x^2+x}\right)y = \frac{x(x+1)^2}{x^2+x}$$

$$P(x) = \frac{2-x^2}{x^2+x}$$

Now $-\int P(x)\,dx = -\int \frac{2-x^2}{x^2+x}\,dx$

$$= \int \left(\frac{x^2-2}{x^2+x}\right)dx$$

$$= \int \left(\frac{x^2+x-x-2}{x^2+x}\right)dx$$

$$= \int 1 - \frac{(x+2)}{x^2+x}\,dx$$

$$= \int (1 - \left(\frac{x+2}{x(x+1)}\right))\,dx$$

$$= \int (1 - \left[\frac{2}{x} - \frac{1}{x+1}\right])\,dx$$

$$\frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$= \int \left(1 - \frac{2}{x} + \frac{1}{x+1}\right)dx$$

$x + 2 = A(x+1) + Bx$

$x = 0 \rightarrow A = 2$

$x = -1 \rightarrow B = -1$

$= x - 2\log x + \log(x+1)$

$= x - \log x^2 + \log(x+1)$

$= x + \log(x+1) - \log x^2$

$$= x + \log \frac{x+1}{x^2}$$

$$v = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx$$

$$= \int \frac{1}{(e^x)^2} e^{x+\log\left(\frac{x+1}{x^2}\right)} dx$$

$$= \int \frac{1}{e^{2x}} e^x . e^{\log\left(\frac{x+1}{x^2}\right)} dx$$

$$= \int \frac{1}{e^x} . \frac{x+1}{x^2} dx$$

$$= \int \frac{1}{e^x} \left(\frac{1}{x} + \frac{1}{x^2} \right) dx$$

$$= \int e^{-x} \frac{1}{x} dx + \int e^{-x} \frac{1}{x^2} dx$$

$$= x^{-1}.(-e^{-x}) + \int e^{-x} \left(-\frac{1}{x^2} \right) dx + \int \frac{e^{-x}}{x^2} dx$$

$u = \frac{1}{x} = x^{-1}$ $\quad$ $dv = e^{-x}\,dx$

$$= \frac{-e^{-x}}{x} - \int \frac{e^{-x}}{x^2} dx + \int \frac{e^{-x}}{x^2} dx$$

$du = -x^{-2}$ $\quad$ $v = -e^{-x}$

$$v = -\frac{1}{x} e^{-x}$$

$\therefore$ $y_2 = vy_1$

$$= -\frac{1}{x} e^{-x} e^x$$

$$y_2 = -\frac{1}{x}$$

$\therefore$ The solution is $y = c_1 e^x - c_2 \frac{1}{x}$

To find the particular solution.

The particular solution is $y = v_1 y_1 + v_1 y_2$

$$R(x) = \frac{x(x+1)^2}{x^2 + x}$$

$$= \frac{x(x+1)^2}{x(x+1)}$$

R(x) = x + 1

$y_1 = e^x$ $\qquad y_2 = -\frac{1}{x} = -x^{-1}$

$y_1' = e^x$ $\qquad y_2' = x^{-2} = \frac{1}{x^2}$

W (y_1, y_2) = y_1 y_2' - y_1' y_2

$$= e^x . \frac{1}{x^2} - e^x \left(-\frac{1}{x} \right)$$
$$= e^x \left(\frac{1}{x^2} + \frac{1}{x} \right)$$

$$W(y_1, y_2) = e^x \left(\frac{1+x}{x^2} \right) \neq 0$$

$$v_1 = -\int \frac{R(x)\, y_2}{W(y_1, y_2)} dx$$

$$= -\int \frac{(x+1)(-\frac{1}{x})}{e^x (1+\frac{x}{x^2})} dx$$

$$= \int \frac{(x+1)}{e^x} . \frac{x^2}{x+1} dx$$

$$= \int e^{-x}\, x\, dx$$

u = x, dv = e^{-x}dx
du = dx, v = - e^{-x}

$$= -xe^{-x} + \int e^{-x}\, dx$$

$$= -xe^{-x} - e^{-x}$$

$$= -e^{-x}(x+1)$$

$$v_2 = \int \frac{R(x)\, y_1}{W(y_1, y_2)} dx$$

$$= \int \frac{(x+1)e^{x}}{e^{x}\left(\frac{1+x}{x^{2}}\right)}dx$$

$$= \int x^{2}\, dx$$

$$= \frac{x^{3}}{3}$$

The particular solution is

$y_p = v_1 y_1 + v_2 y_2$

$$= (-e^{-x}(x+1))e^{x} + \frac{x^{3}}{3}\left(-\frac{1}{x}\right)$$

$$= -(x+1) - \frac{x^{2}}{3}$$

$\therefore$ The complete solution is

$y = c_1 y_1 + c_2 y_2 + v_1 y_1 + v_2 y_2$

$$= c_1 e^{x} - c_2 \frac{1}{x} - (x+1) - \frac{x^{2}}{3}$$

$$= c_1 e^{x} - c_2 x^{-1} - x - 1 - \frac{1}{3}x^{2}$$

A Review of Power Series:

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^{n} = a_0 + a_1 x + a_2 x^{2} + \ldots \qquad \text{.....(1)}$$

is called a power series in x

The series $\sum_{n=0}^{\infty} a_n (x - x_0)^{n} = a_0 + a_1(x - x_0) + a_2(x - x_0)^{2} + \ldots.$ .....(2)

is a power series in $(x - x_0)$

The series equation (1) is said to converge at the point x if the limit $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ is exist and in this case the sum of the series is the value of this limit.

Let $\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \ldots$ be a series of non-zero constant.

Clearly at x = 0, the series equation (1) is convergent.

We are interested in other points at which the series is convergent.

For this we use the Ratio test. Which states that " $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = L$ exist then the series Σ u_n converges if L < 1 and diverges if L > 1"

We may identify Σ $a_n x^n$ with Σ u_n $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{a_{n+1}\, x^{n+1}}{a_n\, x^n}\right|$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x|$$

$$\therefore \underset{n\to\infty}{Lt} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{Lt} \left|\frac{a_{n+1}}{a_n}\right| |x|$$

The converges depend upon the value of x.

Let $R = \underset{n\to\infty}{Lt} \left|\frac{a_n}{a_{n+1}}\right|$

We've $L = \underset{n\to\infty}{Lt} \left|\frac{u_{n+1}}{u_n}\right|$

$$= \underset{n\to\infty}{Lt} \left|\frac{a_{n+1}}{a_n}\right| |x|$$

$$= \frac{1}{R} |x|$$

The series is convergent if L < 1

$$\therefore \frac{1}{R} |x| < 1$$

$\rightarrow |x| < R$

Also the series is diverges if L > 1

$\therefore \frac{1}{R}|x| > 1$

$\rightarrow |x| > R$

Each power series in x has the radius of convergence where $0 \le R \le \infty$ with the property that the series converges if $|x| < R$ and diverges if $|x| > R$.

Also if R = 0, then no x satisfies $|x| < R$ and if $R = \infty$, then no x satisfies $|x| > R$.

If R is finite and non-zero then it determines an interval of convergence are $-R < x < R$ such that inside the interval the series converges and outside the interval it diverges.

$\therefore$ Power series may or may not converge at either n points of its interval of convergence.
Using Power series to find the Taylor's series.

Suppose that $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ Converges for $|x| < R$ with $R > 0$

Denote its sum by f(x)

$\therefore f(x) = \sum_{n=0}^{\infty} a_n x^n$

$= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots$

Then f(x) is continuous and has derivatives for all orders for $|x| < R$.

The series can be differentiated term wise

$\therefore f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots$

$f''(x) = 2a_2 + 3.2a_3x + 4.3.a_4x^2 + 5.4.a_5x^3 + \ldots$

$f'''(x) = 3.2.a_3 + 4.3.2.a_4x + 5.4.3.a_5x^2 + \ldots$

$f^{(IV)}(x) = 4.3.2.a_4 + 5.4.3.2.a_5x + \ldots$

⋮ ⋮

Put $x = 0$ in the above

$f(0) = a_0$

$f'(0) = a_1$

$$f''(0) = 2a_2 \rightarrow a_2 = \frac{f''(0)}{2!}$$

$f'''(0) = 6a_3$

$$\rightarrow a_3 = \frac{f'''(0)}{2.3}$$

$$a_3 = \frac{f'''(0)}{3!}$$

$f^{IV}(0) = 4.3.2.\ a_4$

$$a_4 = \frac{f^{IV}(0)}{4!}$$

$\vdots \qquad \vdots$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$\vdots \qquad \vdots$

The series $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$

$$\therefore f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(0)}{(n+1)!}x^{n+1} + \ldots.$$

This is known as the Taylors series for $f(x)$

$$\therefore f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots. + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

Where $R_n(x)$ is called the remainder after n-terms.

Also $$f(x) = f(x_0) + \frac{f'(x_0)}{1!}.(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$$

This is known as the Taylor's series for $f(x)$ at $x = x_0$

Note:

Suppose $f(x) = a_0 + a_1x + a_2x^2 + \ldots + \ldots$ is convergent in $-R < x < R$ ($|x| < R$) then $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots\ldots$ is also convergent in $-R < x < R$.

i) If $f(x) = g(x)$, then $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, …

ii) $f(x) \pm g(x) = (a_0 \pm b_0) + (a_1 \pm b_13x^2) + (a_2 \pm b_2)\, x^2 + \ldots$

iii) $f(x) \,.\, g(x) = \Sigma\, c_nx^n$, where $c_n = a_0b_n + a_1\, b_{n-1} + \ldots + a_n\, b_0$

$f(x)\, g(x)$ are also converges in the same interval $-R < x < R$.

Algebraic and Transcendental function:

An algebraic function is a polynomial, a rational function (or) more generally any function $y = f(x)$ that satisfies an equation of the form

$$p_n(x)\, y^n + p_{n-1}(x)\, y^{n-1} + \ldots + p_1(x)\, y + p_0(x) = 0$$

Where each $p_i(x)$ is a polynomial.

All other functions which do not satisfy a polynomial equation of the above form are called Transcendental function.

Eg:

i) Polynomials are algebraic functions.

ii) e^x, $\log x$ are transcendental functions.

Definition: (Elementary Function)

A Combination of (a) addition, subtraction, multiplication, devition, logarithmic function, or forming functions of functions) algebraic and transcendental function is called the elementary function.

Eg:

$$y = \tan\left(\frac{xe^{\frac{1}{x}} + \tan^{-1}(1+x^2)}{\sin x \cos 2x - \sqrt{\log x}}\right)^{\frac{1}{2}}$$

Some standard series:

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$

2. $\sin x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

4. $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

5. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Problem:

It is well known from elementary algebra that $1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$ if $x \neq 1$.

Use this to show that the expansions $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ and $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ are valid for $|x| < 1$. Applying the latter to show that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ and $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ for $|x| < 1$.

Solution:

Given that $1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$ if $x \neq 1$

For $|x| < 1$

$$\lim_{n\to\infty} x^n = \lim_{n\to\infty} x^{n+1} = 0$$

$$\lim_{n\to\infty} (1 + x + x^2 + \ldots + x^n) = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x}$$

$$1 + x + x^2 + \ldots = \frac{1}{1-x}$$

$$\therefore \frac{1}{1-x} = 1 + x + x^2 + \ldots \qquad \ldots\ldots (1)$$

Sub -x for x

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \ldots. \qquad \ldots\ldots. (2)$$

Integrating equation (2)

$$\int \frac{1}{1-x}dx = \int (1-x+x^2-x^3+x^4-x^5...)dx$$

$$\log(1+x) = A+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+....$$

Put x = 0

$$\text{Log}(1) = A$$

$$A = 0$$

$$\therefore \log(1+x) = x$$

Sub x^2 for x in equation (2)

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8...$$

Integrating

$$\int \frac{1}{1+x^2}dx = \int (1-x^2+x^4-x^6+x^8-....)dx$$

$$\tan^{-1}(x) = B+x+\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...$$

Put x = 0

$$\tan^{-1}(0) = B$$

$$\Rightarrow B = 0$$

$$\therefore \tan^{-1}(x) = B+x+\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...$$

Problem:

Show that the series $y=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+...$ converges for all x an verify that it is a solution of $xy''+y'+xy = 0$.

Solution

Given $y=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+...$

$$u_n = \frac{(-1)x^{2n}}{2^2.4^2.6^2...(2n)^2}$$

$$u_n = \frac{(-1)^{n+1}x^{2n+2}}{2^2.4^2.6^2...(2n+2)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{(-1)^{n+1}x^{2n+2}}{2^2.4^2.6^2...(2n+2)^2} \times \frac{2^2.4^2.6^2....(2n)^2}{(-1)^n x^{2n}}$$

$$= \frac{-x^2}{(2n+2)^2}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{x^2}{(2n+2)^2}$$

$$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{Lt}\frac{x^2}{(2n+2)^2}$$

$$\underset{n\to\infty}{Lt}\left|\frac{u_{n+1}}{u_n}\right| \to 0 \quad \text{for all } x$$

$\therefore \sum u_n$ is convergent

(OR)

$$a_n = \frac{(-1)^n}{2^2.4^2.6^2...(2n)^2}$$

$$a_{n+1} = \frac{(-1)^{n+1}}{2^2.4^2.6^2...(2n+2)^2}$$

$$\frac{a_n}{a_{n+1}} = \frac{(-1)^n}{2^2.4^2...(2n)^2} \times \frac{2^2.4^2.6^2...(2n+1)^2}{(-1)^{n+1}}$$

$$= -(2n+2)^2$$

$$R = \underset{n\to\infty}{Lt}\left|\frac{a_n}{a_{n+1}}\right| = \underset{n\to\infty}{Lt}(2n+2)^2 \to \infty$$

$\therefore$ Radius of convergence $R = \infty$

$\therefore$ The series of convergent for all x

$$y=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\dots$$

$$y'=-\frac{2x^2}{2^2}+\frac{4x^3}{2^2.4^2}-\frac{6x^5}{2^2.4^2.6^2}+\dots$$

$$y''=-\frac{2}{2^2}+\frac{4.3.x^3}{2^2.4^2}-\frac{6.5.x^4}{2^2.4^2.6^2}+\dots$$

$$xy''=-\frac{2x}{2^2}+\frac{4.3.x^3}{2^2.4^2}-\frac{6.5.x^5}{2^2.4^2.6^2}+\dots$$

$$xy''+y'=\left(-\frac{2x}{2^2}+\frac{4.3.x^3}{2^2.4^2}-\frac{6.5.x^5}{2^2.4^2.6^2}+\dots\right)+\left(-\frac{2x}{2^2}+\frac{4x^3}{2^2.4^2}-\frac{6x^5}{2^2.4^2.6^2}+\dots\right)$$

$$=\frac{-2x}{2^2}(1+1)\frac{4x^3}{2^2.4^2}(3+1)-\frac{6x^5}{2^2.4^2.6^2}(5+1)+\dots$$

$$xy''+y'=-x+\frac{x^2}{2^2}-\frac{x^5}{2^2.4^2}+\frac{x^6}{2^2.4^2.6^2}+\dots$$

$$=-x\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}.+\dots\right)$$

$$= -xy$$

$\therefore xy''+y'+xy = 0$

Problem:

Use the expansion $\frac{1}{1-x}=1+x+x^2+x^3+\dots$To find the Power Series for $\frac{1}{(1-x)^2}$

(a) By squaring b) By differentiating

Solution:

Given $\frac{1}{1-x}=1+x+x^2+x^3+\dots$

a) By Squaring

$$\left(\frac{1}{1-x}\right)^2=\left(1+x+x^2+x^3+\dots\right)\left(1+x+x^2+x^3+\dots\right)$$

$$\frac{1}{(1-x)^2} = 1 + x + x^2 + x^3 + \ldots + x + x^2 + x^3 + x^4$$

$$+ \ldots x^2 + x^3 + x^4 + \ldots x^3 + x^4 + x^5 + \ldots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots.$$

b) By differentiating

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \ldots.$$

Diff, $\frac{(1-x)(0) - 1(-1)}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

Series solutions of first order Equations:

Solve y' = y

Solution:

The series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

(i.e) $y = a_0 + a_1 x_1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$

y' = y

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots.. = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

Equating the like coefficients

$a_1 = a_0$

$2a_2 = a_1$

$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$3a_3 = a_2$

$a_3 = \frac{a_2}{3} = \frac{a_0}{2.3}$

$4a_4 = a_3$

$a_4 = \frac{a_3}{4} = \frac{a_0}{2.3.4}$

$5a_5 = a_4$

$a_5 = \frac{a_4}{5} = \frac{a_0}{2.3.4.5}$

$y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots..$

$= a_0 + a_0 x + \frac{a_0}{2}x^2 + \frac{a_0}{2.3}x^3 + \frac{a_0}{2.3.4}x^4 + \frac{a_0}{2.3.4.5}x^5 + \ldots.$

$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right)$

$y = a_0e^x$

Verification:

$y' = y$

$\frac{dy}{dx} = y$

$\frac{dy}{y} = dx$

Integrating

$\int \frac{dy}{y} = \int dx$

$\log y = x+\log c$

$= \log e^x+\log c$

$\log y = \log ce^x$

$y = ce^x$

Problem

Solve $y = (1+x)^p$ where p is a arbitrary constant, and y(0) = 1.

Solution:

$y = (1+x)^p$ …….. (1)

Diff (1) w.r.t x

$y' = p(1+x)^{p-1}$

$y' = \frac{p(1+x)^p}{(1+x)}$

$(1+x)y' = p(1+x)^p$

$(1+x)y' = py$

$y'+xy' = py$ …….. (2)

To find the solution of (2)

The series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$

$y' = a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots$

$xy' = a_1x+2a_2x^2+3a_3x^3+4a_4x^4+\ldots$

$py = pa_0+pa_1x+pa_2x^2+pa_3x^3+pa_4x^4+\ldots$

(2) ⇒

$(a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots)+(a_1x+2a_2x^2+3a_3x^3+4a_4x^4+\ldots)$

$= pa_0+pa_1x+pa_2x^2+pa_3x^2+pa_3x^3+pa_4x^4+\ldots$

Equating the like coeff

$a_1 = pa_0$ $[\because y(0) = a_0 = 1]$

$a_1 = p.1$

$a_1 = p$

$2a_2+a_1 = pa_1$

$2a_2 = pa_1-a_1$

$$a_2 = \frac{a_1(p-1)}{2}$$

$$a_2 = \frac{p(p-1)}{2}$$

$3a_3+2a_2 = pa_2$

$3a_3 = pa_2-2a_2$

$3a_3 = a_2(p-2)$

$$a_3 = \frac{(p-2)}{3}.\frac{p(p-1)}{2}$$

$$a_3 = \frac{p(p-1)(p-2)}{2.3}$$

$4a_4+3a_3 = pa_3$

$4a_4 = pa_3-3a_3$

$4a_4 = (p-3)a_3$

$$a_4 = \frac{(p-3)}{2}.\frac{p(p-1)(p-2)}{2.3}$$

$$a_4 = \frac{p(p-1)(p-2)(p-3)}{2.3.4}$$

$\vdots \quad \vdots$

$y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$

$$y = 1+px+\frac{p(p-1)}{2!}x^2+\frac{p(p-1)(p-2)}{3!}x^3+\frac{p(p-1)(p-2)(p-3)}{4!}x^4+\ldots$$

(i.e)

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \ldots$$

for $|x| < 1$

This expansion is called binomial series.

Problem

Express $\text{Sin}^{-1}x$ in the form of power series $\Sigma a_n x^n$ by solving $y' = (1-x^2)^{-1/2}$ in two ways use the result to obtain the formula

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}.\frac{1}{3.2^3} + \frac{1.3}{2.4} + \frac{1}{5.2^5} + \frac{1.3.5}{2.4.6}.\frac{1}{7.2^7} + \ldots$$

Solution

$y' = (1-x)^{-1/2}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$dy = \frac{dx}{\sqrt{1-x^2}}$$

$$\int dy = \int \frac{dx}{\sqrt{1-x^2}}$$

$y = \sin^{-1}(x) + C$ …….. (1)

$y' = (1-x^2)^{-1/2}$

$$y' = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}(x^2)^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}(x^2)^3 + \ldots$$

$$y' = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots$$

$$\frac{dy}{dx} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots$$

$$dy = \left(1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots\right)dx$$

$$\int dy = \int\left(1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\ldots\right)dx$$

$$y \quad = A+x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+\ldots \qquad \ldots\ldots. (2)$$

Equating (1) & (2)

$$\sin^{-1}(x) + C = A+x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+\ldots$$

Put x = 0

sin-1(0) + C = A

0 + C = A

A = C

We get,

$$\sin^{-1}(x) = x+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+\ldots \qquad \ldots\ldots.. (3)$$

Put x = ½ in equation (3)

$$\sin^{-1}(1/2) = \frac{1}{2}+\frac{1}{2}.\frac{(1/2)^3}{3}+\frac{1.3}{2.4}.\frac{(1/2)^5}{5}+\frac{1.3.5}{2.4.6}.\frac{(1/2)^7}{7}+\ldots$$

$$\frac{\pi}{6} = \frac{1}{2}+\frac{1}{2}.\frac{1}{3.2^3}+\frac{1.3}{2.4}.\frac{1}{5.2^5}+\frac{1.3.5}{2.4.6}.\frac{1}{7.2^7}+\ldots$$

Problem

Given ordinary non-linear equation y' = 1+y^2. The differential equations consider in the text and proceeding problem are all linear. The equation y' = 1+ y^2 is non-linear and it is easy to see directly that y = tanx is the particular solution for which y(0) = 0. Show that $\tan x = x+\frac{1}{3}x^3+\frac{2}{5}x^5+\ldots$ By assuming a solution for the above equation y' = 1 + y^2 in the form of a power series $\Sigma a_n x^n$ and finding the a_n's. By differentiating the equation y' = 1+y^2 repeatedly to obtain y'' = 2yy', y''' = 2yy''+2$(y')^2$ and using the formula $an = \frac{f^{(n)}(0)}{n!}$

Solution

Given $y' = 1 + y^2$

$$\frac{dy}{dx} = 1+y^2$$

$$\frac{dy}{1+y^2} = dx$$

∫ing

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1}(y) = x + C$$

Put x = 0 and y(0) = 0

$$\tan^{-1}(0) = 0 + C$$

$$C = 0$$

$$\tan^{-1}(y) = x$$

$$\Rightarrow y = \tan x \qquad \ldots\ldots\ldots (1)$$

The series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \quad y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$$

$$y' = a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots$$

$$y' = 1+y^2$$

$$a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4\ldots\ldots = 1+(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)^2$$

$$= 1+a_0^2+a_1^2x^2+a_2^2x^4+a_3^2x^6+\ldots + 2a_0a_1x+2a_0a_2x^2+2a_0a_3x^3+\ldots + 2a_1a_2x^3+2a_1a_3x^4+\ldots+2a_2a_3x^5+\ldots$$

Equating the like coefficient

$a_1 = 1 + a_0^2$ $\qquad$ ($\because y(0) = a_0 = 0$)

$a_1 = 1$

$2a_2 = 2a_0a_1$ $\qquad$ ($\because a_0 = 0, a_1 = 1$)

$a_2 = 0$

$3a_3 = a_1^2+2a_0a_2$

$3a_3 = 1 + 2(0)$

$a_3 = 1/3$

$4a_4 = 2a_0a_3 + 2a_1a_2$ $(\because a_0 = 0, a_2 = 0)$

$4a_4 = 2(0) + 2(1)(0)$

$a_4 = 0$

$5a_5 = a_2^2 + 2a_1a_3$

$5a_5 = 0 + 2(1)(1/3)$

$5a_5 = 2/3$

$a_5 = 2/15$

$y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots$

$$= 0+1.x+0.x^2+\frac{1}{3}x^3+0.x^4+\frac{2}{15}x^5+\ldots$$

$$y = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots \qquad \ldots\ldots. (2)$$

From (1) & (2)

$$\tan x = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots$$

Given $y' = 1+ y^2$

Differentiating

$y'' = 2yy'$

$y''' = 2yy''+2y'y'$

$y''' = 2yy''+2(y')^2$

$y^{(iv)} = 2yy'''+2y'y''+4y'y''$

$y^{(iv)} = 2yy'''+6y'y''$

$y^{(v)} = 2yy^{(iv)}+2y'y'''+6y''y'''+6y'y'''$

$y^{(v)} = 2yy^{(iv)} + 6(y'')^2 + 8y'y'''$

. .

. .

Let $y(x) = f(x)$

$y'(x) = f'(x)$

$y''(x) = f''(x)$

$y'''(x) = f'''(x)$

$y(0) = 0$

$f(0) = 0$

$f'(0) = y'(0)$

$y' = 1++y^2$

$y'(0) = 1 + [y(0)]^2$

$= 1 + 0$

$y'(0) = 1$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$a_1 = \frac{f^1(0)}{1!}$$

$$a_1 = \frac{1}{1!} = 1$$

$$a_2 = \frac{f''(0)}{2!}$$

$f''(0) = y''(0) = 2y(0)y'(0)$

$= 2(0)(1)$

$y''(0) = 0$

$a_2 = 0$

$$a_3 = \frac{f'''(0)}{3!}$$

$f'''(0) = y'''(0)$

$= 2y(0)y''(0)+2(y'(0))^2 = 2(0)(0) + 2(1)^2$

$f'''(0) = 2$

$a_3 = \frac{2}{3!} = \frac{2}{2.3}$

$a_3 = \frac{1}{3}$

$a_4 = \frac{f^{(iv)}(0)}{4!}$

$f^{(iv)}(0) = y^{(iv)}(0)$

$= 2y(0)y'''(0)+6y'(0)y''(0) = 2(0)(2) +6(1)(0)$

$f^{(iv)}(0) = 0$

$a4 = \frac{0}{4!} = 0$

$a_4 = 0$

$a_5 = \frac{f^{(v)}(0)}{5!}$

$f^{(v)}(0) = y^{(v)}(0)$

$= 2y(0)y^{(iv)}(0)+6(y''(0))^2+8y'(0)y'''(0)$

$= 2(0)(0) + 6(0)^2+ 8(1)(2)$

$f^{(v)}(0) = 16$

$a_5 = \frac{f^{(v)}0}{5!} = \frac{16}{1.2.3.4.5} = \frac{2}{15}$

$\therefore y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots$

$= 0+1.x+0.x^2+\frac{1}{3}x^3+0.x^4+\frac{2}{15}x^5+\ldots$

$y = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots$

UNIT II

Second order linear Equations (Ordinary points)

Consider the homogenous linear equations of the second order y'' + p(x)y'+q(x) = 0. The solution of this equation depends upon the nature of functions p(x) and Q(x). If these functions are analytic at $x = x_0$. Then the power series solution of the above point $x = x_0$ exist and coverage at $x = x_0$. The points at which P(x) and Q(x) are analytic are called ordinary points of the equations.

The point at which these functions are not analytic is called singular points.

Problem

Slove y'' + y = 0

Solution

Gn y'' + y = 0

Here P(x) = 0, Q(x) = 1

P(x) and Q(x) are analytic at all points.

The series solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

ie) $y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\ldots$

$y' = a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+\ldots$

$y'' = 2a_2+3.2.a_3x+4.3.a_4x^2+5.4.a_5x^3+6.5.a_6x^4+\ldots$

$y'' = -y$

$2a_2+3.2.a_3x+4.3.a_4x^2+5.4.a_5x^3+6.5.a_6x^4+\ldots = -a_0-a_1x-a_2x^2-a_3x^3-a_4x^4-a_5x^5-a_6x^6-\ldots$

Equating the like coefficient

$2a_2 = -a_0$

$$a_2 = \frac{-a_0}{2} = \frac{-a_0}{2!}$$

$2.3.a_3 = -a_1$

$$a_3 = \frac{-a_1}{2.3} = \frac{-a_1}{3!}$$

$4.3.a_4 = -a_2$

$$a_4 = \frac{-a_2}{4.3}$$

$$a_4 = \frac{-1}{4.3}\left(\frac{-a_1}{2!}\right)$$

$$a_4 = \frac{a_0}{4!}$$

$5.4.a_5 = -a_3$

$$5.4.a_5 = -\left(\frac{-a_1}{3!}\right)$$

$$a_5 = \frac{a_1}{3!4.5} = \frac{a_1}{5!}$$

$6.5.a_6 = -a_4$

$$6.5.a_6 = -\left(\frac{a_0}{4!}\right)$$

$$a_6 = -\frac{a_0}{4!5.6} = -\frac{a_0}{6!}$$

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The series solution is

$$y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\ldots$$

$$= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \frac{-a_0}{6!}x^6 - ....$$

$$= a_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...\right)+a_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-...\right)$$

$= a_0\cos x + a_1\sin x$

$\therefore \quad y = a_0\cos x + a_1\sin x$

Where $y_1 = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ and $y_2 = \cos x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

(OR)

Also, $y = \Sigma a_n x^n$

$y' = \Sigma n a_n x^{n-1}$

$y'' = \Sigma n(n-1) a_n x^{n-2}$

Sub in $y'' + y = 0$

$\Sigma n(n-1) a_n x^{n-2} + \Sigma a_n x^n = 0$

$\Sigma (n+2)(n+2-1) a_{n+2} x^{n+2-2} + \Sigma a_n x^n = 0$

$\Sigma (n+2)(n+1) a_{n+2} x^n + \Sigma a_n x^n = 0$

$((n+2)(n+1) a_{n+2} + a^n)\, x^n = 0$

Equating the coeff of x^n to zero

$\therefore (n+2)(n+1) a_{n+2} = -a_n$

$$a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$$

Put n = 0

$$\Rightarrow a_2 = \frac{-a_0}{1(2)} = \frac{-a_0}{2!}$$

Put n = 1

$$\Rightarrow a_3 = \frac{-a_1}{2.3} = \frac{-a_1}{3!}$$

Put n = 2

$$\Rightarrow a_4 = \frac{-a_2}{3.4} = -\left(\frac{-a_0}{2!}\right).\frac{1}{3.4}$$

$$= \frac{a_0}{4!}$$

Problem

Solve the Legendre's eqn

An equation is of the form $(1-x^2)y''-2xy'+P(P+1)y = 0$ is called the legendre's equation, Where P is a constant.

Solution

The legendre's equation is

$$(1-x^2)y''-2xy'+P(P+1)y = 0 \quad \ldots\ldots. (1)$$

$$\Rightarrow y''-\frac{2x}{1-x^2}y'+\frac{P(P+1)}{1-x^2}y = 0$$

$$P(x) = \frac{-2x}{1-x^2},\ Q(x) = \frac{P(P+1)}{1-x^2}$$

Clearly P(x) and Q(x) are analytic

∴ The series solution is

$$y = \Sigma a_n x^n$$

$$y' = \Sigma n a_n x^{n-1}$$

$$y'' = \Sigma n(n-1)a_n x^{n-2}$$

Sub in (1)

$$(1-x^2)(\Sigma n(n-1)a_n x^{n-2}) - 2x\Sigma n a_n x^{n-1} + P(P+1)\Sigma a_n x^n = 0$$

$$\Sigma n(n-1)a_n x^{n-2} - \Sigma n(n-1)a_n x^n - 2\Sigma n a_n x^{n-1} + P(P+1)\Sigma a_n x^n = 0$$

$$\Sigma(n+2)(n-1)a_{n+2}x^n - \Sigma n(n-1)a_n x^n - 2\Sigma n a_n x^n + P(P+1)\Sigma a_n x^n = 0$$

$$\{(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + P(P+1)a_n\}x^n = 0$$

Equating the coeff of x^n to zero.

$$(n+2)(n+1)a_{n+2} - [n(n-1)+2n - P(P+1)]a_n = 0$$

$(n+2)(n+1)a_{n+2} - [n^2-n+2n-P^2-P]a_n = 0$

$(n+2)(n+1)a_{n+2} - [n^2+n-P^2-P]a_n = 0$

$(n+2)(n+1)a_{n+2} = [n^2+n-P^2-P]a_n$

$= [(n+p)(n-p)+(n-p)]\ a_n$

$$\therefore a_{n+2} = \frac{(n-P)(n+P+1)}{(n+1)(n+2)} a_n \qquad \ldots\ldots (2)$$

Put n = 0

$$a_2 = \frac{(-P)(P+1)}{1.2} a_0$$

Put n = 1

$$a_3 = \frac{(1-P)(P+2)}{2.3} a_1$$

Put n = 2

$$a_4 = \frac{(2-P)(P+3)}{3.4} a_2$$

$$= \frac{(2-P)(P+3)}{3.4} . \frac{(-P)(P+1)}{1.2} a_0$$

$$a_4 = \frac{-P(P+1)(2-P)(P+3)}{4!} . a_0$$

Put n = 3

$$a_5 = \frac{(3-P)(P+4)}{4.5} a_3$$

$$= \frac{(3-P)(P+4)}{4.5} . \frac{(1-P)(P+2)}{2.3} a_1$$

$$a_5 = \frac{(1-P)(3-P)(P+2)(P+4)}{5!} a_1$$

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The solution is

$$y = \Sigma a_n x^n$$

ie) $$y = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\ldots$$

$$= a_0 + a_1 x - \frac{P(P+1)}{2!}a_0x^2 + \frac{(1-P)(P+2)}{3!}a_1x^3 - \frac{P(P+1)(2-P)(P+3)}{4!}a_0x^4 + \frac{(1-P)(3-P)(P+2)(P+4)}{5!}a_1x^5 - \ldots\ldots$$

$$= a_0\left[1 - \frac{P(P+1)}{2!}x^2 - \frac{P(P+1)(2-P)(P+3)}{4!}x^4 - \ldots\ldots\right] + a_1\left[x - \frac{(P-1)(P+2)}{3!}x^3 + \frac{(P-1)(P-3)(P+2)(P+4)}{5!}x^5 + \ldots\ldots\right]$$

$\therefore y = a_0y^1 + a_1y^2$

Where $y_1 = 1 - \frac{P(P+1)}{2!}x^2 - \frac{P(P+1)(2-P)(P+3)}{4!}x^4 - \ldots\ldots$ and

$$y_2 = x - \frac{(P-1)(P+2)}{3!}x^3 + \frac{(P-1)(P-3)(P+2)(P+4)}{5!}x^5 + \ldots..$$

Clearly y_1 and y_2 are linearly independent

From eqn (1) & (2)

$$a_{n+2} = \frac{(n-P)(n+P+1)}{(n+1)(n+2)}a_n$$

$$\Rightarrow \frac{a_n}{a_{n+2}} = \frac{(n+1)(n+2)}{(n-P)(n-P+1)}$$

$$\frac{a_n}{a_{n+2}} = \frac{n^2(1+1/n)(1+2/n)}{n^2(1-P/n)(1-P/n+1/n)}$$

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+2}}\right| = \lim_{n\to\infty}\left|\frac{(1+1/n)(1+2/n)}{(1-P/n)(1-P/n+1/n)}\right|$$

$$= 1$$

$$R = 1$$

Note:

i. The solution y_1 and y_2 are in the form of infinite series, but generally it is not an elementary function. They are called Legendre's functions are valid for $|x| < 1$.
ii. If P is a +ive even integer, the series for y_1 terminate at a particular stage and y_1 becomes a polynomial, y_2 still remains an infinite series.
iii. If P is a +ive odd integer, the series for y_2 terminate at a particular stage and y_2 becomes a polynomial and y_1 still remains an infinite series.
iv. The polynomial defined in Note (ii) & (iii) are called Legendre's Polynomial.
v. For different values of P we get different Legendre's equation.

Theorem

Let x_0 be an ordinary point of the differential equation $y''+P(x)y'+Q(x)y = 0$ and let a_0 and a_1 be arbitrary constants. Then there exist a unique function y(x) that is analytic at x_0, is a solution of the given equation in a certain neighborhood of this point and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Further more if the power series expansion of P(x) and Q(x) are valid on an interval $|x-x_0| < R$, $R > 0$. Then the power series expansion of this solution is also valid on the same interval.

Proof

It is enough. if we prove the theorem for the point $x_0 = 0$

Given that $y'' + P(x)y' + Q(x)y = 0$ ……..(1)

The functions P(x) and Q(x) are analytic at the point $x = x_0$.

We have assume that P(x) & Q(x) are analytic at the origin

∴ The power series expansion

$$P(x) = \sum_{n=0}^{\infty} p(x)x^n = p_0 + p_1x + p_2x^2 + \ldots$$

$$Q(x) = \sum_{n=0}^{\infty} q(x)x^n = q_0 + q_1x + q_2x^2 + \ldots$$

To find the solution for $y'' + P(x)y' + Q(x)y = 0$ in the form of the power series $y = \Sigma a_n x^n$

ie) $y = a_0+a_1x+a_2x^2+a_3x^3+\ldots$

$y = a_1+2a_2x+3a_3x^3+\ldots$

∴ $y(0) = a_0$

y'(0) = a_1

Which corresponds to the given condition $y(x_0) = a_0$, $y'(x_0) = a_1$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Sub the above in (1)

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \left[\sum_{n=0}^{\infty} p(x) x^n\right]\left[\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right] + \left[\sum_{n=0}^{\infty} q(x) x^n\right]\left[\sum_{n=0}^{\infty} a_n x^n\right] = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} (p_{n-k}(k+1) a_{k+1}) x^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} (q_{n-k} a_k) x^n = 0$$

$$\sum_{n=0}^{\infty} \left\{(n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} p_{n-k}(k+1) a_{k+1} + \sum_{k=0}^{n} q_{n-k} a_k\right\} x^n = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} p_{n-k}(k+1) a_{k+1} + \sum_{k=0}^{n} q_{n-k} a_k = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} \left[(k+1) p_{n-k} a_{k+1} + q_{n-k} a_k\right] = 0$$

$$\therefore (n+2)(n+1) a_{n+2} = -\sum_{k=0}^{n} \left[(k+1) p_{n-k} a_{k+1} + q_{n-k} a_k\right] \quad \ldots\ldots\ldots (2)$$

Put n = 0 in (2)

$$\therefore 2.1.a_2 = -\sum_{k=0}^{n=0} \left[(k+1) p_{n-k} a_{k+1} + q_{n-k} a_k\right]$$

$$2.a_2 = -[1.p_0 a_1 + q_0 a_0]$$

$$a_2 = \frac{-[1.p_0 a_1 + q_0 a_0]}{2}$$

Put n = 1 in (1)

$$\therefore 3.2.a_3 = -\sum_{k=0}^{n=1} \left[(k+1)p_{n-k}a_{k+1} + q_{n-k}a_k\right]$$

$$2.3.a_3 = -[1.p_{1-0}a_1 + q_{1-0}a_0 + (1+1)p_{1-1}a_{1+1} + q_{1-1}a_1]$$

$$2.3.a_3 = -[p_1 a_1 + q_1 a_0 + 2p_0 a_2 + q_0 a_1]$$

$$2.3.a_3 = -[p_1 a_1 + q_1 a_0 + 2p_0\left(\frac{-(p_0 a_1 + q_0 a_0)}{2}\right) + q_0 a_1]$$

$$= -[p_1a_1+q_1a_0-p_0^2a_1-p_0q_0a_0+q_0a_1]$$

$$= -p_1a_1-q_1a_0-p_0^2a_1+p_0q_0a_0-q_0a_1$$

$$2.3\ a_3 = a_0(p_1q_0-q_1)+a_1(p_0^2-p_1-q_0)$$

$$a_3 = \frac{a_0(p_0q_0 - q_1) + a_1(p_0^2 - p_1 - q_0)}{2.3}$$

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∴ we get $a_2, a_3, a_4, \ldots\ldots$ interms of a_0 and a_1

ie) All the coefficients of the series $\sum_{n=0}^{\infty} a_n x^n$ in terms of a_0 and a_1.

Hence the solution of the equation $y''+P(x)y'+Q(x)y = 0$ exist in the form of the series $y = \sum_{n=0}^{\infty} a_n x^n$

Problem

The equation $y''+(p+\frac{1}{2}-\frac{1}{4}x^2)y = 0$, where p is constant. Certainly has a series solution of the form $y = \Sigma a_n x^n$.

a) S.T. the coefficient an are relate by the three term recursion formula $(n+1)(n+2)a_{n+2}+\left(p+\frac{1}{2}\right)a_n-\frac{1}{4}a_{n-2}=0$.

b) If the independent variable is change from y to ω by means of $y = \omega e^{x^2/4}$, Show that the equation is transformed into $\omega''-x\omega'+p\omega = 0$.

c) Verify that the equation in (b) has a two term recursion formula and find its general soln.

Solution

a) Given $y''+(p+\frac{1}{2}-\frac{1}{4}x^2)y = 0$ …… (1)

Here P(x) = 0, Q(x) = $p+\frac{1}{2}-\frac{1}{4}x^2$.

∴ P(x) and Q(x) are analytic.
∴ we assume the power series solution

$y = \Sigma a_n x^n$

$y' = \Sigma n a_n x^{n-1}$

$y'' = \Sigma n(n-1)a_n x^{n-2}$

sub in (1)

$$\sum n(n+1)a_n x^{n-2}+\left(p+\frac{1}{2}-\frac{1}{4}x^2\right)\sum a_n x^n = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n+\left(p+\frac{1}{2}\right)\sum a_n x^n-\frac{1}{4}\sum a_n x^{n+2} = 0$$

$$(n+2)(n+1)a_{n+2}x^n+\left(p+\frac{1}{2}\right)a_n x^n-\frac{1}{4}a_n x^{n+2} = 0$$

$$\left((n+2)(n+1)a_{n+2}+\left(p+\frac{1}{2}\right)a_n-\frac{1}{4}a_{n-2}\right)x^n = 0$$

$$(n+2)(n+1)a_{n+2}+\left(p+\frac{1}{2}\right)a_n-\frac{1}{4}a_{n-2} = 0$$

b) $y = \omega e^{x^2/4}$

$$y' = \omega e^{-x^2/4}.\frac{(-2x)}{4} + e^{-x^2/4}\omega'$$

$$= \frac{-2}{4}\omega x e^{-x^2/4} + e^{-x^2/4}\omega'$$

$$y'' = -\frac{1}{2}\omega x e^{-x^2/4}.\frac{(-2x)}{4} + \omega e^{-x^2/4}(1) + xe^{-x^2/4}\omega' + e^{-x^2/4}\frac{(-2x)}{4}\omega' + e^{-x^2/4}.\omega''$$

$$= \frac{1}{4}x^2\omega e^{-x^2/4} - \frac{1}{2}\omega e^{-x^2/4} - \frac{1}{2}x\omega' e^{-x^2/4} - \frac{1}{2}x\omega' e^{-x^2/4} + \omega'' e^{-x^2/4}$$

$$y'' = \frac{1}{4}x^2\omega e^{-x^2/4} - \frac{1}{2}\omega e^{-x^2/4} - x\omega' e^{-x^2/4} + \omega'' e^{-x^2/4}$$

sub in (1)

$$y'' + (p + \frac{1}{2} - \frac{1}{4}x^2)y = 0$$

$$\frac{1}{4}x^2\omega e^{-x^2/4} - \frac{1}{2}\omega e^{-x^2/4} - x\omega' e^{-x^2/4} + \omega'' e^{-x^2/4} + (p + \frac{1}{2} - \frac{1}{4}x^2)\omega e^{-x^2/4} = 0$$

$$\frac{1}{4}x^2\omega e^{-x^2/4} - \frac{1}{2}\omega e^{-x^2/4} - x\omega' e^{-x^2/4} + \omega'' e^{-x^2/4} + p\omega e^{-x^2/4} + \frac{1}{2}\omega e^{-x^2/4} - \frac{1}{4}x^2\omega e^{-x^2/4} = 0$$

$$\Rightarrow \omega'' e^{-x^2/4} - x\omega' e^{-x^2/4} + p\omega e^{-x^2/4} = 0$$

$$\Rightarrow e^{-x^2/4}(\omega'' - x\omega' + p\omega) = 0$$

$$\because e^{-x^2/4} \neq 0$$

$$\Rightarrow \omega'' - x\omega' + p\omega = 0$$

c) Given $\omega'' - x\omega' + p\omega = 0$

$P(x) = -x$ and $Q(x) = p$

⇒ P(x) and Q(x) are analytic

∴ We assume power series solution

$\omega = \Sigma a_n x^n$

$\omega' = \Sigma n a_n x^{n-1}$

$\omega'' = \Sigma n(n-1)a_n x^{n-2}$

sub in the given equation

$\Sigma n(n-1)a_n x^{n-2} - x\Sigma n a_n x^{n-1} + p\Sigma a_n x^n = 0$

$\Sigma(n+2)(n-1)a_{n+2}x^n - \Sigma n a_n x^n + p\Sigma a_n x^n = 0$

$(n+2)(n-1)a_{n+2}x^n - n a_n x^n + p a_n x^n = 0$

$(n+2)(n-1)a_{n+2}x^n - (n-p)a_n x^n = 0$

$[(n+2)(n-1)a_{n+2} - (n-p)a_n]x^n = 0$

$\Rightarrow (n+2)(n-1)a_{n+2} - (n-p)a_n = 0$

$\Rightarrow (n+2)(n-1)a_{n+2} = (n-p)a_n$

$$\Rightarrow a_{n+2} = \frac{n-p}{(n+2)(n+1)}a_n$$

Put n = 0

$$a_2 = \frac{-p}{1.2}a_0$$

Put n = 1

$$a_3 = \frac{1-p}{2.3}a_1$$

Put n = 2

$$a_4 = \frac{2-p}{3.4}a_2$$

$$= \frac{2-p}{3.4}\left(\frac{-p}{1.2}\right)a_0$$

$$= \frac{-p(2-p)}{1.2.3.4}a_0$$

$$a_4 = \frac{p(p-2)}{4!}a_0$$

Put n = 3

$$a_5 = \frac{3-p}{4.5} a_3$$

$$= \frac{3-p}{4.5}\left(\frac{1-p}{2.3}\right)a_1$$

$$a_5 = \frac{(p-1)(p-3)}{5!} a_1$$

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∴ The solution is

$\omega = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots$

$$\omega = a_0 + a_1 x - \frac{p}{2!}a_0x^2 + \frac{1-p}{3!}a_1x^3 + \frac{p(p-2)}{4!}a_0x^4 + \frac{(p-1)(p-3)}{5!}a_1x^5 + \ldots$$

$$= a_0 + a_1 x - \frac{p}{2!}a_0x^2 - \frac{p-1}{3!}a_1x^3 + \frac{p(p-2)}{4!}a_0x^4 + \frac{(p-1)(p-3)}{5!}a_1x^5 + \ldots$$

$$= a_0\left[1 - \frac{p}{2!}x^2 + \frac{p(p-2)}{4!}x^4 + \ldots\right] + a_1\left[x - \frac{p-1}{3!}x^3 + \frac{(p-1)(p-3)}{5!}x^5 - \ldots\right]$$

$\omega = a_0y_1+a_2y_2$

Where $y_1 = 1 - \frac{p}{2!}x^2 + \frac{p(p-2)}{4!}x^4 + \ldots$ and $y_2 = x - \frac{p-1}{3!}x^3 + \frac{(p-1)(p-3)}{5!}x^5 - \ldots$

Regular Singular Points

Consider the homogeneous linear equation of second order $y''+P(x)y'+Q(x)y = 0$ …. (1). The solution of the equation depends upon the nature of the functions P(x) and Q(x). If these functions are analytic at the Point x = 0 then the points are called the **ordinary points** of the equation.

The points at which the functions are not analytic is called **singular points**.

A singular points x_0 of the equation (1) is said to be **regular** if the functions $(x-x_0)P(x)$ and $(x-x_0)^2P(x)$ are analytic.

If these functions are not analytic then x_0 is called irregular singular point.

Problem

Locate and classify its singular points on the x axis

a) $x^3(x-1)y''-2(x-1)y'+3xy = 0$

b) $x^2(x^2-1)^2y''-x(1-x)y'+2y = 0$

Solution

b) Gn : $x^2(x^2-1)^2y''-x(1-x)y'+2y = 0$

$$\Rightarrow y''-\frac{x(1-x)}{x^2(x^2-1)^2}y'+\frac{2}{x^2(x^2-1)^2}y=0$$

$$P(x)=\frac{x(1-x)}{x^2(x^2-1)^2}$$

$$=\frac{(1-x)}{x(x+1)^2(x-1)^2}$$

$$P(x)=\frac{1}{x(x+1)^2(x-1)}$$

$$Q(x) = \frac{2}{x^2(x^2-1)^2}$$

$$= \frac{2}{x^2(x+1)^2(x-1)^2}$$

Here P(x) and Q(x) are not analytic At x = 0

$$x.P(x)=\frac{x}{x(x+1)^2(x-1)}$$

$$= \frac{1}{(x+1)^2(x-1)}$$

$$x^2Q(x)=\frac{2x^2}{x^2(x+1)^2(x-1)^2}$$

$$= \frac{2}{(x+1)^2(x-1)^2}$$

$\therefore$ x = 0 is a regular singular point.

At x = 1

$$(x-1).P(x) = \frac{x-1}{x(x+1)^2(x-1)}$$

$$= \frac{1}{x(x+1)^2}$$

$$= \underset{x \to x}{lt} \frac{1}{x(x+1)^2} \quad = \quad \frac{1}{4}$$

$$(x-1)Q(x) = \frac{2(x-1)^2}{x^2(x+1)^2(x-1)^2}$$

$$= \frac{2}{x^2(x+1)^2}$$

$$= \underset{x-1}{lt} \frac{2}{x^2(x+1)^2} \quad = \quad \frac{2}{4} = \frac{1}{2}$$

$\therefore$ x = 1 is a regular singular point at pt x = -1

$$(x+1).P(x) = \frac{x+1}{x(x+1)^2(x-1)}$$

$$= \frac{1}{x(x+1)(x-1)}$$

$$= \underset{x \to x}{lt} \frac{1}{x(x+1)(x-1)}$$

$$= \frac{1}{-1(0)(-2)}$$

$$= \infty$$

$$(x+1)^2 Q(x) = \frac{2(x+1)^2}{x^2(x+1)^2(x-1)^2}$$

$$= \frac{2}{x^2(x-1)^2}$$

$$= \lim_{x \to -1} \frac{2}{x^2(x-1)^2} \quad = \quad \frac{2}{4} \; = \; \frac{1}{2}$$

∴ x = -1 is a irregular singular point.

Problem

Determine the nature of pt x = 0, for each of the following equation.

a) $y'' + (\sin x).y = 0$

b) $x^3 y'' + (\sin x)y = 0$

c) $x^4 y'' + (\sin x)\, y = 0$

Solution

a) Gn : $y'' + (\sin x).y = 0$

$\Rightarrow$ P(x) = 0; Q(x) = sinx

At pt x = 0

P(x) and Q(x) are analytic
∴ x =0 is an ordinary pt.

b) Gn : $x^3 y'' + (\sin x)y = 0$

$$\Rightarrow y'' + \frac{\sin x}{x^3} y = 0$$

P(x) = 0, Q(x) = $\dfrac{\sin x}{x^3}$

At pt x = 0

P(x) and Q(x) are not analytic

x. P(x) = 0, $\quad x^2 Q(x) = \dfrac{x^2 \sin x}{x^3}$

$$\underset{x \to 0}{Lt}\, x^2 Q(x) = \underset{x \to 0}{Lt} \frac{\sin x}{x^2} = 1$$

xP(x) and x^2Q(x) are analytic at pt x = 0

∴ x = 0 is a regular singular pt.

c) Gn : x^4y''+(sinx) y = 0

$$\Rightarrow y'' + \frac{\sin x}{x^4} y = 0$$

P(x) = 0, Q(x) = $\dfrac{\sin x}{x^4}$

Here P(x) and Q(x) are not analytic at x = 0

At x = 0

x. P(x) = 0

$$x^2 Q(x) = \frac{x^2 \sin x}{x^4}$$

$$= \frac{\sin x}{x^2}$$

$$= \underset{x \to 0}{Lt} \frac{\sin x}{x^2}$$

∴ x = 0 is irregular singular point.

Frobenius Method

Consider a differential equation y''+P(x)y'+Q(x)y = 0. If x = 0 is an ordinary point we can get independent solutions in the form of the power series $\sum_{n=0}^{\infty} a_n x^n$

If x =0 is a regular singular point of the equation then a solution of the form $\sum_{n=0}^{\infty} a_n x^n$ may not be possible. In such a cases the series solution can be obtained by the method of Frobenius series.

We can take the series solution as $y = x^m \sum_{n=0}^{\infty} a_n x^n$ where m is a constant to determined.

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n$

$= x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots\ldots)$

$= a_0x^m + a_1x^{m+1} + a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+a_5x^{m+5}+\ldots\ldots$

$y' = ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+\ldots\ldots$

$y'' = m(m-1)a_0x^{m-2} +m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m+(m+2)(m+3)a_3x^{m+1}+\ldots\ldots$

Sub these in the equation $y''+P(x)y'+Q(x)y = 0$ and equating the coefficient of varius power to 0. Equating the lowest power of x to zero. We get, a quadratic equation in m.

This equation is called the indicial equation of the given differential equation. The roots of this equation m_1& m_2 (say) are called the exponents of the differential equation.

If m_1 and m_2 are distinct, then there are two independent solutions.

If $m_1 = m_2$, there is only one independent solution say y1.

The other solution may be obtained by $y_2 = \nu y_1$, where $\nu = \int \frac{1}{{y_1}^2} e^{-\int P(x)dx}\, dx$

Problem

Solve : $2x^2y''+x(2x+1)y'-y = 0$

Solution

Gn : $2x^2y''+x(2x+1)y'-y = 0$ ……..... (1)

$\Rightarrow\ y''+\frac{x(2x+1)}{2x^2}y'-\frac{1}{2x^2}y = 0$

$P(x) = \frac{x(2x+1)}{2x^2}$, $Q(x) = -\frac{1}{2x^2}$

$\Rightarrow P(x) = \frac{(2x+1)}{2x}$, $Q(x) = -\frac{1}{2x^2}$

Here P(x) and Q(x) are not analytic at x = 0

$x.\ \ P(x) = \frac{x.(2x+1)}{2x}$

$$= \lim_{x\to 0} \frac{(2x+1)}{2} \quad = \frac{1}{2}$$

$$x^2Q(x) = -\frac{1}{2x^2}$$

$$= \lim_{x\to 0}\left(-\frac{1}{2}\right)$$

∴ x = 0 is a regular singular point.

∴ The series soln is

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n$

$= x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots)$

$= a_0x^m + a_1x^{m+1} + a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\dots$

$y' = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\dots$

$y'' = m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} + (m-1)(m+2)a_2x^m+ (m+2)(m+3)a_3x^{m+1} + (m+4)(m+3)\, a_4x^{m+2}+ \dots$

Sub in eqn (1)

$2x^2[m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m+ (m+2)(m+3)a_3x^{m+1} + (m+4)(m+3)\, a_4x^{m+2}+ \dots\dots]+x(2x+1)\,[ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\dots] -[a_0x^m +a_1x^{m+1} +a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\dots] = 0$

Equating the coefficient of Lower power of x

ie) x^m to zero

$2m(m-1)a_0+ma_0-a_0 = 0$

$(2m(m-1)+m-1)a_0 = 0$

$\Rightarrow (2m^2-2m+m-1)a_0 = 0$

$\Rightarrow 2m^2-2m-1 = 0$ $\qquad (\because a_0\neq 0$

This is called the indicial equation

$2m^2-2m-1 = 0$

$2m^2-2m+m-1 = 0$

$2m(m-1)+1(m-1) = 0$

$(m-1)(2m+1) = 0$

$\Rightarrow m = 1, 2m = -1$

$\Rightarrow m_1 = 1, m_2 = -1/2$

Equating the coefficient of x^{m+1} to zero

$2m(m+1)a_1+2ma_0+(m+1)a_1-a_1 = 0$

$[2m(m+1)+(m+1)-1]\, a_1 = -2ma_0$

$(2m^2+2m+m+1-1)a_1 = -2ma_0$

$\Rightarrow (2m^2+3m)\, a_1 = -2ma_0$

$\Rightarrow m(2m+3)a_1 = -2ma_0$

$$\Rightarrow a_1 = \frac{-2}{2m+3}.a_0$$

Equating the coefficient of x^{m+2} to zero

$2(m+1)(m+2)a_2+2(m+1)a_1+(m+2)a_2 - a_2 = 0$

$[2(m+1)(m+2)+(m+2)-1]a_2 = -2(m+1)a_1$

$[2(m+1)(m+2)+(m+1)]\, a_2 = -2(m+1)a_1$

$(m+1)\, [2(m+2)+1]a_2 = -2(m+1)a_1$

$\Rightarrow (2m+4+1)a_2 = -2a_1$

$\Rightarrow (2m+5)a_2 = -2a_1$

$$(2m+5)a_2 = -2\left(\frac{-2}{2m+3}.a_0\right)$$

$$a_2 = \frac{2^2}{(2m+3)(2m+5)}.a_0$$

Equating the coefficient of x^{m+3} to zero

$2(m+2)(m+3)a_3+2(m+2)a_2+(m+3)a_3-a_3 = 0$

$(2(m+2)(m+3)m+3-1)a_3 = -2(m+2)a_2$

$(2(m+2)(m+3)+(m+2))a_3 = -2(m+2)a_2$

$(m+2)\,[2(m+3)+1]\,a_3 = -2(m+2)a_2$

$(2m+6+1)a_3 = -2a_2$

$$(2m+7)a_3 = -2\left(\frac{2^2}{(2m+3)(2m+5)}.a_0\right)$$

$$a_3 = -\frac{2^3}{(2m+3)(2m+5)(2m+7)}.a_0$$

. .
. .
. .

Put $m_1 = 1$

$$\therefore a_1 = \frac{-2}{2(1)+3}.a_0 = \frac{-2}{5}a_0$$

$$a_2 = \frac{2^2}{(2+3)(2+5)}.a_0 = \frac{2^2}{5\times 7}a_0$$

$$= \frac{2^2}{35}a_0$$

$$a_3 = \frac{-2^3}{(2+3)(2+5)(2+7)}a_0$$

$$= \frac{-2^3}{5\times 7\times 9}a_0$$

. .
. .
. .

Put $m_2 = -1/2$

$$a_1 = \frac{-2}{2\left(\frac{-1}{2}\right)+3}a_0$$

$$a_1 = \frac{-2}{2} a_0$$

$a_1 = -a_0$

$$a_2 = \frac{2^2}{\left[2\left(\frac{-1}{2}\right)+3\right]\left[2\left(\frac{-1}{2}\right)+5\right]}.a_0$$

$$a_2 = \frac{2^2}{2\times 4} a_0$$

$$a_2 = \frac{1}{2} a_0$$

$$a_3 = \frac{-2^3}{\left[2\left(\frac{-1}{2}\right)+3\right]\left[2\left(\frac{-1}{2}\right)+5\right]\left[2\left(\frac{-1}{2}\right)+7\right]} a_0$$

$$a_3 = \frac{-2^3}{2\times 4\times 6} a_0$$

$$a_3 = \frac{-1}{6} a_0$$

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The series solution is

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$= x^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots)$$

$$= x\left[a_0 - \frac{2}{5} a_0 x + \frac{2^2}{35} a_0 x^2 - \frac{2^3 x^3}{315} a_0 + \ldots\ldots.\right]$$

Put $a_0 = 1$

$$y_1 = x\left[1 - \frac{2}{5} x + \frac{2^2}{35} x^2 - \frac{2^3 x^3}{315} + \ldots\ldots.\right]$$

$$y_2 = x^{m_2}\sum_{n=0}^{\infty} a_n x^n$$

$= x^{-1/2}(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots)$

$$= x^{-1/2}\left[a_0 - a_0x + \frac{1}{2}a_0x^2 - \frac{1}{6}a_0x^3 + \ldots\ldots.\right]$$

$$= x^{-1/2}a_0\left[1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\ldots.\right]$$

Put $a_0 = 1$

$$= x^{-1/2}\left[1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\ldots.\right]$$

These two solutions are linearly independent.

∴ The general solution is $y = c_1y_1+c_2y_2$

$$\therefore y = C_1x\left[1 - \frac{2}{5}x + \frac{2^2}{35}x^2 - \frac{2^3}{315}x^3 + \ldots\ldots.\right] + C_2x^{-1/2}\left[1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\ldots.\right]$$

Bessel's Equation

An equation is of the form $x^2y''+xy'+(x^2-P^2)y = 0$ where P is a constant is called the Bessel's equation.

Problem

When P = 0, the Bessel's equation becomes x2y''+xy'+x2y = 0. Show that it indicial equation has only one root, and deduce that $y = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}.x^{2n}$ is the corresponding Frobenius series solution.

Solution

Gn : $x^2y''+xy'+x^2y = 0$. ……. (1)

$\Rightarrow \quad y''+\frac{1}{x}y'+y = 0$

$P(x) = \frac{1}{x}$, Q(x) = 1

P(x) is not analytic and Q(x) is analytic

∴ x = 0 is not an ordinary pt.

At x = 0

$$xP(x) = \frac{x}{x} = 1$$

$x^2Q(x) = x^2$

At the pt x = 0, xP(x) and $x^2Q(x)$ are analytic

∴ x = 0 is a regular singular pt

∴ The Frobenius series solution is

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n$

$= x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots)$

$= a_0x^m + a_1x^{m+1} + a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\ldots\ldots$

$y' = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\ldots\ldots$

$y'' = m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} + (m-1)(m+2)a_2x^m+ (m+2)(m+3)a_3x^{m+1}$
$+ (m+4)(m+3)\, a_4x^{m+2}+ \ldots\ldots$

Sub in eqn (1)

$x^2[m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} + (m-1)(m+2)a_2x^m+ (m+2)(m+3)a_3x^{m+1}$
$+ (m+4)(m+3)\, a_4x^{m+2}+ \ldots]+x[ma_0x^{m-1} + (m+1)a_1x^m$
$+(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\ldots]$
$+x^2[a_0x^m + a_1x^{m+1} + a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\ldots] = 0$

Equating the coeff of x^m to zero

$m(m-1)a_0+ma_0 = 0$

$(m^2-m+m)a_0 = 0$

$m^2 a_0 = 0$

∵ $a_0 \neq 0$, $m^2 = 0$

$m = 0$

Equating the coeff of x^{m+1} to zero

$m(m+1)a_1+(m+1)a_1 \quad = 0$

$(m+1)a_1 (m+1) \quad = 0$

$a_1(m+1)^2 \quad = 0$

$\Longrightarrow a_1 = 0 \qquad (\because m = 0)$

Equating the coeff of x^{m+2} to zero

$(m+1)(m+2)a_2+(m+2)a_2+a_0 \quad = 0$

$(m+2)a_2[m+1+1]+a_0 \quad = 0$

$(m+2)^2a_2 \quad = -a_0$

$$a_2 = \frac{-a_0}{(m+2)^2} = \frac{-a_0}{2^2} \qquad (\because m = 0)$$

Equating the coeff of x^{m+3} to zero

$(m+2)(m+3)a_3+(m+3)a_3+a_1 \quad = 0$

$(m+3)a_3(m+2+1)+a_1 \quad = 0$

$(m+3)a_3(m+3) \quad = -a_1$

$a_3 (m+3)^2 \quad = 0$

$a_3 \quad = 0 \qquad (\because m = 0$

Equating the coeff of x^{m+4} to zero

$(m+3) (m+4) a_4+(m+4)a_4+a_2 = 0$

$(m+4)a_4(m+3+1) \quad = -a_2$

$(m+4)^2a_4 \quad = -(-a_0/2^2)$

$$a_4 = \frac{a_0}{2^2(m+4)^2}$$

$$a_4 = \frac{a_0}{2^2.4^2}$$

.

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∴ The series solution is

$$y = x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots)$$

$$= x_0(a_0 + 0.x - \frac{-a_0}{2^2}x^2 + 0.x^3 + \frac{a_0}{2^2.4^2}x^4 + ...)$$

$$y = a_0 - \frac{a_0}{2^2}x^2 + \frac{a_0}{2^2.4^2}x^4 - \frac{a_0}{2^2.4^2.6^2}x^6 + ...)$$

$$y = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + ...\right)$$

Take $a_0 = 1$

$$y = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + ...$$

$$= 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4.(1.2)^2} - \frac{x^6}{2^6.(1.2.3)^2} + ...$$

$$= 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{(2^2)^2.(2!)^2} - \frac{x^6}{22.(1.2.3)^2} + ...$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2^2)^n (n!)^2}$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

Problem

Consider the diff equation $y'' + \frac{1}{x^2}y' - \frac{1}{x}y = 0$

a) Show that x = 0 is an irregular singular pt.
b) Use a fact that $y_1 = x$ is a solution to find the second independent solution y_2.
c) Show that the 2^{nd} solution y_2 found in (b) cannot expressed as a Frobenisious series

Solution

Given : $y'' + \frac{1}{x^2}y' - \frac{1}{x}y = 0$ ……… (1)

a) $P(x) = \frac{1}{x^2}$, $Q(x) = -\frac{1}{x^3}$

Here P(x) and Q(x) are not analytic at the pt x = 0
∴ x = 0 is not an ordinary point

$$x.P(x) = \frac{x}{x^2} = \frac{1}{x}$$

$$x^2.Q(x) = \frac{-x^2}{x^3} = -\frac{1}{x}$$

∴ xP(x) and x^2Q(x) are not analytic at the point x = 0

∴ x = 0 is an irregular singular point.

b) T.P. y_1 =x is the solution of equation (1)

y_1 = x, y_1'=1, y_1''

$$y_1''+\frac{1}{x^2}y_1'-\frac{1}{x^3}y1 = 0+\frac{1}{x^2}(1)-\frac{1}{x^3}(x)$$

$$= \frac{1}{x^2}-\frac{1}{x^2}$$

$$= 0$$

∴ y_1= x is the solution of equation (1)

To find y_2

$y_2 = \nu y_1$ where $\nu = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx$

$$\nu = \int \frac{1}{x^2} e^{-\int \frac{1}{x^2}dx} dx$$

$$= \int \frac{1}{x^2} e^{\frac{1}{x}} dx$$

$$= -\int e^z \, dz \qquad \text{Put } z = 1/x,\ dz = \frac{-1}{x^2}dx$$

$= -e^z$

$$\nu = -e^{\frac{1}{x}}$$

$y_2 = v y_1$

$y_2 = -e^{\frac{1}{x}}.x$

c) Which cannot be expressed in ascending power of x. So it is to a Frobenisius series

Problem

The diff equation $x^2y''+(3x-1)y'+y = 0$ has x = 0 is an irregular singular pt. If the Frobenisius series is inserted into this eqn. Show that m = 0 and the corresponding Frobenisius series solution is the power series $y=\sum_{n=0}^{\infty} n!\,x^n$. Which converges only at x =0. This demonstrate that even when a Frobenisius series formally satisfied such an equation it is not necessarily a valid solution.

Solution

Given : $x^2y''+(3x-1)y'+y = 0$ …… (1)

$$\Longrightarrow y''+\frac{3x-1}{x^2}y'+\frac{1}{x^2}y = 0$$

$$P(x) = \frac{3x-1}{x^2}, \qquad Q(x)=\frac{1}{x^2}$$

Here P(x) and Q(x) are not analytic at x = 0

x P(x) is not analytic

∴ x =0 is a irregular singular point

If $y=x^m\sum_{n=0}^{\infty} a_n x^n$

$= x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$

$= a_0x^m +a_1x^{m+1} +a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\ldots$

$y' = ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\ldots$

$y'' = m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m+ (m+2)(m+3)a_3x^{m+1}$
$+ (m+4)(m+3)\, a_4x^{m+2}+ \ldots$

Sub in eqn (1)

$x^2[m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m$
$+ (m+2)(m+3)a_3x^{m+1}+ (m+4)(m+3)\, a_4x^{m+2}+ \ldots]$
$+(3x-1)[ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}$

$+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+......]$
$+a_0x^m +a_1x^{m+1} +a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+... \qquad = 0$

Equating the coefficient of x^{m-1} to zero

$-ma_0 = 0$

$\because a_0 \neq 0 \quad m = 0$

$m(m-1)a_0+3ma_0-(m+1)a_1+a_0 = 0$

$(m(m-1)+3m+1)a_0 = (m+1) a_1$

$(m+1) a_1 = (m^2-m+3m+1)a_0$

$(m+1) a_1 = (m^2+2m+1)a_0$

$(m+1) a_1 = (m+1)^2a_0$

$a_1 = (m+1) a_0$

$a_1 = a_0$

Equating the coeeficient of x^{m+1} to zero

$m(m+1)a_1+3(m+1)a_1-(m+2)a_2+a_1 = 0$

$a_1(m(m+1)+3(m+1)+1) = (m+2)a_2$

$a_1(m^2+4m+4) = (m+2)a_2$

$a_1(m+2)^2 = a_2$

$2a_1 = a_2 \qquad (m = 0)$

$a_2 = 2a_0$

Equating the coeff of xm+2 to zero

$(m+1)(m+2)a_2+3(m+2)a_2 - (m+3)a3+a_2 = 0$

$[(m+1)(m+2)+3(m+2)+1]a_2 = (m+3)a_3$

$(m2+6m+9)a_2 = (m+3)a_3$

$(m+3) (m+3) a_2 = (m+3)a_3$

$a_3 = a_2(m+3)$

$a_3 \quad = 6\, a_0$

The series solution is

$$y = x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$$

$$= x^0\,(a_0+a_0x+2a_0x^2+6a_3x^3+a_4x^4+\ldots)$$

$$= a_0(1+x+2x^2+6x^3+\ldots\ldots)$$

Put $a_0 = 1$

$$y = 1+x+2x^2+6x^3+\ldots$$

$$= 1+x+2!x^2+3!x^3+4!x^4+\ldots)$$

$$y = \sum_{n=0}^{\infty} n!\,x^n$$

Let us discuss the convergence

$u_n = n!\, x^n, \qquad u_{n+1} = (n+1)!\, x^{n+1}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!\,x^{n+1}}{n!\,x^n}$$

$$= (n+1)\, x$$

$$\left|\frac{u_{n+1}}{u_n}\right| = (n+1)|x|$$

For convergence

$$\lim_{n\to\infty}(n+1)|x| < 1$$

$$|x| < \lim_{n\to\infty}\frac{1}{(n+1)}$$

Hence the series is convergent only for the point x = 0, so that the above series cannot be taken as a valid solution of the differentiable equation.

Problem

The equation $x^2y''-3xy'+(4x+4)y = 0$ has only one Frobenius series solution find it.

Solution

Given : $x^2y''-3xy'+(4x+4)y = 0$ ……….. (1)

$$\Rightarrow y''-\frac{3}{x}y'+\left(\frac{4x+4}{x^2}\right)y = 0$$

$$P(x) = -\frac{3}{x},\ Q(x)=\frac{4x+4}{x^2}$$

At point x = 0

$$xP(x) = -\frac{3x}{x} = -3$$

$$x^2Q(x)=\frac{x^2(4x+4)}{x^2} = 4x+4$$

At the point x = 0

$$xP(x) = -3 = P_0$$

$$x^2Q(x)= 4 \ = q_0$$

The indicial equation is

$m(m-1)+mp_0+q_0 = 0$

$m(m-1)-3m+4 = 0$

$m^2-m-3m+4 = 0$

$m^2-4m+4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

This has only one root

$$y=x^m\sum_{n=0}^{\infty}a_nx^n$$

$= x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$

$= a_0x^m +a_1x^{m+1} +a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\ldots$

$y' \quad = ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\ldots$

$y'' \quad = m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m+ (m+2)(m+3)a_3x^{m+1}$
$+ (m+4)(m+3) a_4x^{m+2}+ \ldots$

Sub in (1)

$x^2[m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m$
$+ (m+2)(m+3)a_3x^{m+1}+ (m+4)(m+3) a_4x^{m+2}+ \ldots]$
$-3x[ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\ldots]$
$+(4x+4)[a_0x^m +a_1x^{m+1} +a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\ldots] \quad = 0$

$x^m[m(m-1)a_0-3ma_0+4a_0]+x^{m+1}[m(m+1)a_1-3(m+1)a_1+4a_0+4a_1]$
$+x^{m+2}[(m+1)(m+2)a_2-3(m+2)a_2+4a_1+4a_2]$
$+x^{m+3}[(m+2)(m+3)a_3-3(m+3)a_3+4a_2+4a_3]+\ldots. \quad = 0$

Equating x^{m+1} to zero

$m(m+1)a_1-3(m+1)a_1+4a_0+4a_1 \quad = 0$

$m(m+1)-3(m+1)+a)a_1+4a_0 \quad = 0$

$(m^2+m-3m-3+4)a_1 \quad = -4a_0$

$(m^2-2m+1)a_1 \quad = -4a_0$

Put m = 2
$(2-1)2a_1 \quad = -4a_0$

$a_1 \quad = -4a_0$

Equating x^{m+2} to zero

$(m+1) (m+2)a_2 - 3(m+2)a_2+4a_1+4a_2 \quad = 0$

$((m+1) (m+2) - 3(m+2)+4)a_2 \quad = -4a_1$

Put m = 2
$[(3 \times 4)-3(4)+4]a_2 \quad = -4(-4a_0)$
$(12-12+4)a_2 \quad = 16a_0$

$4a_2 \quad = 16 a_0$

$a_2 \quad = 4a_0$

Equating x^{m+3} to zero

$(m+2) (m+3)a_3 - 3(m+3)a_3 + 4a_2 + 4a_3 \quad = 0$

$[(m+2)(m+3)-3(m+3)+4a]a_3 \quad = -4a_2$

Put m = 2

$(4\times5 - 3(5) + 4)a_3 \quad = -4(4a_0)$

$(20 - 15 + 4)a_3 = -16\, a_0$

$9\, a_3 \quad = -16\, a_0$

$a_3 \quad = (-16/9)a_0$

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∴ The series solution is

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

$$y = x^2 (a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots)$$

$$= x^2(a_0 - 4a_0x + 4a_0x^2 - \frac{16}{9}a_0x^3 + \ldots\ldots)$$

$$= x^2a_0(1 - 4x + 4x^2 - \frac{16}{9}x^3 - \ldots\ldots)$$

Put $a_0 = 1$

$$y = x^2(1 - 4x + 4x^2 - \frac{16}{9}x^3 - \ldots\ldots)$$

Problem

Find the indicial equation and its roots for each of the following differential equation

a) $x^3y''+(\cos 2x-1)y'+2xy = 0$

Solution

Given : $x^3y''+(\cos 2x-1)y'+2xy = 0$

$$\Rightarrow y'' + \frac{\cos 2x - 1}{x^3}y' + \frac{2}{x^2}y \quad = 0$$

$$P(x) = \frac{\cos 2x - 1}{x^3}, \qquad Q(x) = \frac{2}{x^2}$$

$$xP(x) = \frac{x\cos 2x - 1}{x^3} = \frac{\cos 2x - 1}{x^2} = p_0$$

$$x^2Q(x) = \frac{2x^2}{x^2} = 2 = q_0$$

$$xP(x) = \frac{\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\dots\right) - 1}{x^2}$$

$$= -\frac{2^2}{2!} + \frac{2^4x^2}{4!} - \frac{2^6x^4}{6!} + \dots\dots$$

$$\underset{x\to 0}{Lt}\, xP(x) = -\frac{2^2}{2!} = -2 = p_0$$

The indicial equation is

$$m(m-1) + mp_0 + q_0 = 0$$

$$m(m-1) - 2m + 2 = 0$$

$$m^2-m-2m+2 = 0$$

$$m^2-3m+2 = 0$$

$$m(m-1) - 2(m-1) = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

Problem

Find two independent Frobenius series solution for equation $xy''+2y'+xy = 0$.

Solution

Given : $xy''+2y'+xy = 0$ ………… (1)

$$\Rightarrow y''+\frac{2}{x}y'+y=0$$

$P(x) = \frac{2}{x}$, $Q(x) = 1$

P(x) and Q(x) are not analytic at the pt x = 0

$xP(x) = 2$, $x^2Q(x) = x^2$

At x = 0

$xP(x) = 2 = P_0$, $x2Q(x) = 0 = q_0$

∴ The indicial equation is

$m(m-1)+mp_0+q_0 = 0$

$m^2-m+2m+0 = 0$

$m^2+m = 0$

$m(m+1) = 0$

$m_1 = 0$, $m_2 = -1$

The series solution is

$$y = x^m\sum_{n=0}^{\infty} a_n x^n$$

$= x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$

$= a_0x^m +a_1x^{m+1} +a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\ldots$

$y' = ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\ldots$

$y'' = m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m+ (m+2)(m+3)a_3x^{m+1}$
$+ (m+4)(m+3) a_4x^{m+2}+ \ldots$

Sub in (1)

$x[m(m-1)a_0x^{m-2}+m(m+1)a_1x^{m-1} +(m-1)(m+2)a_2x^m$
$+ (m+2)(m+3)a_3x^{m+1}+ (m+4)(m+3) a_4x^{m+2}+ \ldots]$
$+2[ma_0x^{m-1} +(m+1)a_1x^m +(m+2)a_2x^{m+1}$
$+(m+3)a_3x^{m+2}+(m+4)a_4x^{m+3}+\ldots\ldots]$
$+x[a_0x^m +a_1x^{m+1} +a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\ldots] = 0$

$x^{m-1}[m(m-1)a_0+2ma_0]+x^m[m(m+1)a_1+2(m+1)a_1]$
$+x^{m+1}[(m+1)(m+2)a_2+2(m+2)a_2+a_0]$
$+x^{m+2}[(m+2)(m+3)a_3+2(m+3)a_3+a_1]+\ldots = 0$

Equating the coeff of x^m to zero

$$m(m+1)a1+2(m+1)a_1 = 0$$

$$(m+1)a_1(m+2) = 0$$

$$a_1 = 0$$

Equating coeff of x^{m+1} to zero

$$(m+1)\ (m+2)a_2+2(m+2)a_2+a_0 = 0$$

$$(m+2)a_2[m+1+2]+a_0 = 0$$

$$a_2 = \frac{-a_0}{(m+2)(m+3)}$$

Equating coeff. of x^{m+2} to zero

$$(m+3)\ (m+4)a_4+2(m+4)a_4+a_2 = 0$$

$$(m+4)a_4\ [\ m+3+2] = -a_2$$

$$(m+4)\ (m+5)\ a_4 = \frac{a_0}{(m+2)(m+3)}$$

$$a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}$$

Put m = 0

$a_1 = 0$

$a_2 = \frac{-a_0}{2.3}$

$a_3 = 0$

$a_4 = \frac{a_0}{2.3.4.5}$ and so on

Put m = -1

$a_1 = 0$

$a_2 = \frac{-a_0}{1.2}$

$a_3 = 0$

$a_4 = \frac{a_0}{1.2.3.4}$ and so on

$$y_1 = x^m \sum_{n=0}^{\infty} a_n x^n$$

$y_1 = x^0 (a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots)$

$= a_0 + 0.x - \frac{a_0}{2.3}x^2 + 0.x^3 + \frac{a_0}{2.3.4}x^4 + \ldots..$

$= a_0 - \frac{a_0}{3!}x^2 + \frac{a_0}{5!}x^4 + \ldots\ldots.$

$= a_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\ldots.\right)$

Put $a_0 = 1$

$y_1 = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\ldots.$

$y_1 = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots.\right)$ $= x^{-1} \sin x$

$$y_2 = x^m \sum_{n=0}^{\infty} a_n x^n$$

$y_2 = x^{-1} (a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots\ldots)$

$= x^{-1}\left(a_0 + 0.x - \frac{a_0}{2!}x^2 + 0.x^3 + \frac{a_0}{4!}x^4 + \ldots..\right)$

$= x^{-1}\left(a_0 - \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 - \ldots\ldots.\right)$

$$= x^{-1}a_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+.......\right)$$

Take $a_0 = 1$

$$y_2 \quad = x^{-1}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+.......\right)$$

$y_2 \quad = x^{-1}$ cosx

$\therefore y_1 = x^{-1}$ sinx and $y_2 = x^{-1}$ cosx

Legendre Polynomials

An equation is of the form $(1-x^2)y'' - 2xy' + n(n + 1) y = 0$ is called the Legendre equation where n is a constant.

$$\Rightarrow y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

$$P(x) = \frac{-2x}{1-x^2} \quad Q(x) = \frac{n(n+1)}{1-x^2}$$

$P(x)$ and $Q(x)$ are analytic at point $x = 0$

$\therefore$ The series solution is

$y \;=\; \Sigma a_k x^k$

$y' \;=\; \Sigma a_k x^{k-1}$

$y'' \;=\; \Sigma k(k-1)\, a_k x^{k-2}$

Sub in (1)

$(1 - x^2)\, \Sigma k(k-1)\, a_k x^{k-2} - 2x\Sigma k\, a_k x^{k-1} + n(n + 1)\, \Sigma a_k x^k \quad = 0$

$\Rightarrow k(k-1)\, a_k x^{k-2} - x^2\, k\,(k-1)\, 9k\, x^{k-2} - 2k\, a_k x^k + n(n + 1)\, a_k\, x^k = 0$

$\Rightarrow (k + 2)\,(k + 1)\, a_{k+2} x^t - k(k-1)\, a_k x^k - 2k\, a_k x^k + n(n + 1)\, a_k x^t \quad = 0$

$\Rightarrow \{(k + 2)\,(k + 1)\, a_{k+2} - k(k-1)a_k - 2ka_k + n(n+1)a_k\}\, x^k \quad = 0$

Equating the coeff of x^k to zero

$\therefore (k + 2)(k + 1)a_{k+2} - k(k-1)a_k - 2ka_k + n(n+1)a_k = 0$

$(k + 2)(k + 1)a_{k+2} - [k(k-1) + 2k - n(n+1)]a_k = 0$

$$(k + 2)(k + 1)a_{k+2} = [k^2 - k + 2k - n^2 - n]a_k$$

$$= [k^2 + k - n^2 - n]a_k$$

$$= [(k^2 - n^2) + (k - n)]a_k$$

$$= [(k + n)(k - n) + (k - n)]a_k$$

$$(k + 2)(k + 1)a_{k+2} = (k + n)(k + n + 1)a_k$$

$$\Rightarrow \quad a_{k+2} = \frac{(k-n)(k+n+1)}{(k+1)(k+2)}a_k$$

put $k = k - 2$

$$a_{k-2+2} = \frac{(k-2-n)(k-2+n+1)}{(k-2+1)(k-2+2)}a_{k-2}$$

$$a_k = \frac{(k-n-2)(k+n-1)}{(k-1)(k)}a_{k-2}$$

$$a_k = \frac{-(n-k-2)(n+k-1)}{k(k-1)}a_{k-2}$$

$$\therefore a_{k-2} = \frac{-k(k-1)}{(n-k+2)(n+k-1)}a_k$$

w.k.t $P_n(x)$ is a polynomial of degree n that contains only even or odd powers of x according as n is even or n is odd.

$\therefore$ It can be written as

$P_n(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \ldots$ where the sum ends with a_0 if n is even and $a_1 x$ if n is odd.

Let us find a_{n-2}, a_{n-4}, a_{n-6} interms of a_n

we've $\quad a_{k-2} = \dfrac{-k(k-1)}{(n-k+2)(n+k-1)}a_k$

Put $k = n$

$$a_{n-2} = \frac{-n(n-1)}{(n-n+2)\,(n+n-1)} a_n$$

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

Put k = n – 2

$$a_{n-4} = \frac{-(n-2)\,(n-3)}{(n-(n-2)+2)\,(n+n-2+1)} a_{n-2}$$

$$a_{n-4} = \frac{-(n-2)\,(n-3)}{4(2n-3)} a_{n-2}$$

$$a_{n-4} = \frac{n(n-1)\,(n-2)(n-3)}{2\,.\,4(2n-1)(2n-3)} a_n$$

Put k = n – 4

$$a_{n-6} = \frac{-(n-4)\,(n-5)}{(n-(n-4)+2)\,(n+n-4-1)} a_{n-4}$$

$$= \frac{-(n-4)\,(n-5)}{6\,(2n-5)} a_{n-4}$$

$$a_{n-6} = \frac{-n(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)}{2.4.6.\,(2n-1)\,(2n-3)(2n-5)} a_n$$

etc….

$$\therefore P_n(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + a_{n-6}x^{n-6} + \ldots$$

$$= a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2.\,4.\,(2n-1)\,(2n-3)} a_n x^{n-4}$$

$$- \frac{n(n-1)(n-2)\,(n-3)\,(n-4)\,(n-5)}{2\,.4\,.6(2n-1)\,(2n-3)\,(2n-5)} a_n x^{n-6} + \ldots$$

$$P_n(x) = a_n\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2.\,4.\,(2n-1)\,(2n-3)}\,{}_n x^{n-4}\right.$$

$$-\frac{n(n-1)(n-2)\,(n-3)\,(n-4)\,(n-5)}{2\,.4\,.6(2n-1)\,(2n-3)\,(2n-5)}\,x^{n-6}\ +\ldots\Bigg]$$

Where $a_n = \dfrac{(2n)!}{(n!)^2\,2^n}$

Rodrigues formula

$$P_n(x) = \frac{(2n)!}{(n!)^2\,2^n}\left[x^n - \frac{n(n-1)}{2(2n-1)}\,x^{n-2} + \frac{n(n-1)\,(n-2)\,(n-3)}{2.\,4.\,(2n-1)\,(2n-3)}\,{}_n x^{n-4}\right.$$

$$\left.-\frac{(-1)^k\,n(n-1)(n-2)\ldots\,(n-(2k-1))}{2^k\,k!\,.(2n-1)\,(2n-3)\ldots(2n-(2k-1))}\,x^{n-2k}\right] \quad \text{….. (1)}$$

∴ The coeff of x^{n-2k} in (1) is

$$\frac{(-1)^k\,n(n-1)(n-2)\ldots\,(n-2k+1)}{2^k\,k!\,.(2n-1)\,(2n-3)\ldots(2n-2k+1)} \quad \text{….. (2)}$$

Now, n(n-1) (n-2) … (n – 2k +1) = $\dfrac{\mathrm{n(n-1)\,(n-2)\ldots(n-2k+1)\,(n-2k)!}}{\mathrm{(n-2k)!}}$

$$\mathrm{n(n\text{-}1)\,(n\text{-}2)\ldots(n-2k+1)} = \frac{n!}{\mathrm{(n-2k)!}} \quad \text{….. (3)}$$

$$(2n-1)\,(2n-3)\ldots(2n-2k+1) = \frac{\begin{array}{c}(2n-2k+1)\,(2n-2k+2)\,(2n-2k+3)\ldots..(2n-2k+1)\\(2n-4)\,(2n-3)\,(2n-2)\,(2n-1)\,(2n)\end{array}}{2n(2n-2)\,(2n-4)\ldots(2n-2k+2)}$$

$$= \frac{(2n-2k)!\,(2n-2k+1)\,(2n-2k+2)\ldots(2n-1)(2n)}{(2n-2k)!\,2^k\,n(n-1)(n-2)\ldots(n-k+1)}$$

$$= \frac{(2n)!\,(n-k)!}{(2n-2k)!\,2^k\,(n-k)!\,(n-k+1)\,(n-k+2)\ldots(n-1)n}$$

$$= \frac{(2n)!\,(n-k)!}{(2n-2k)!\,2^k\,n!} \quad \text{….. (4)}$$

Sub in equation (3) & (4) in (2)

The coeff of x^{n-2k} in (1) is

$$= (-1)^k\,\frac{n!}{(n-2k)!}\times\frac{(2n-2k)!\,2^k\,n!}{2^k\,k!\,(2n)!\,(n-k)!}$$

$$= \frac{(-1)^k \ (n!)^2 \ (2n-2k)!}{(2n)!\,k!\,(n-2k)!\,(n-k)!}$$

∴ Equation (1) can be written as

$$P_n(x) = \sum_{k=0} \frac{(2n)!}{(n!)^2\, 2^n} \cdot \frac{(-1)^k (n!)^2 (2n-2k)!\, x^{n-2k}}{k!\,(2n)!\,(n-2k)!\,(n-k)!}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k \ (2n-2k)!}{2^n\, k!\,(n-2k)!\,(n-k)!}\, x^{n-2k}$$

where [n/2] is the usual symbol for the greatest integer ≤ n/2

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \ (2n-2k)!}{2^n\, k!\,(n-2k)!\,(n-k)!}\, x^{n-2k}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^n\, k!\,(n-k)!\,!} \frac{d^n}{dx^n}\left(x^{2n-2k}\right)$$

$$= \frac{1}{2^n\, n!} \frac{d^n}{dx^n} \sum_{k=0}^{[n/2]} \frac{n!\left(x^2\right)^{n-k}}{k!\,(n-k)!} (-1)^k$$

If we extend the range of sum by letting k vary from 0 to n. Which changes nothing. Since the new terms are of degree < n and the nth degrees are zero.

$$\therefore P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} \sum_{k=0}^{n} \binom{n}{k} \left(x^2\right)^{n-k} (-1)^k$$

and the binomial formula yields

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} \sum_{k=0}^{n} \left(x^2 - 1\right)^n$$

This expression for $P_n(x)$ is called Rodrigues formula.

Generating Functions of the Legendre Polynomial:

$$\text{P. T } \frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x)t + P_2(x)t^2 + \ldots + P_n(x)t^n$$

$$= \sum_{n=0}^{\infty} P_n(x) t^n$$

Proof

We've $\dfrac{1}{\sqrt{1-2xt+t^2}} = \left(1-2xt+t^2\right)^{-1/2}$

$= [1- t(2x-t)]^{-1/2}$

$$=1+\frac{1}{2}t(2x-t)+\frac{1/2\ .\ 3/2}{1.2}t^2(2x-t)^2+$$
$$\frac{1/2\ .3/2\ .\ 5/2}{1.2.3}t^3(2x-t)^3+......+$$
$$\frac{1/2\ .3/2\ .\ 5/2.......(2n-5)/2}{1.2....(n-2)}t^{n-2}(2x-t)^{n-2}+$$
$$\frac{1/2\ .\ 3/2\ .5/2\ ...\left(\frac{2n-3}{2}\right)t^{n-1}}{1.2....(n-1)}(2x-t)^{n-1}+$$
$$\frac{1/2\ .\ 3/2\ .5/2\ ...\left(\frac{2n-1}{2}\right)}{1.2....(n-1)}t^n(2x-t)^n+.....$$

$$\text{coeff of } t^n = \frac{1/2\ .\ 3/2\ .5/2\ ...\left(\frac{2n-1}{2}\right)(2x)^n}{1.2.3...n}-\frac{1/2\ .\ 3/2\ .5/2\ ...\left(\frac{2n-3}{2}\right)(n-1)C_1(2x)^{n-2}}{1.2.3...(n-1)}$$
$$+\frac{1/2\ .\ 3/2\ .5/2\ ...\left(\frac{2n-5}{2}\right)(n-2)C_2(2x)^{n-4}}{1.2.3...(n-2)}+.....$$

$$=\frac{1.3.5.....(2n-1)}{2^n\,n!}\ .\ 2^n x^n-\frac{1.3.5.....(2n-3)\,(n-1)\,2^{n-2}x^{n-2}}{2^{n-1}(n-1)!}$$
$$+\frac{1.3.5.......(2n-5)}{2^{n-2}(n-2)!}\,\frac{(n-2)(n-3)}{1.2}2^{n-4}x^{n-4}.......$$

$$=\frac{1.3.5.....(2n-1)}{n!}\ x^n-\frac{1.3.5.....(2n-3)\,n(n-1)\,x^{n-2}}{2.n!}$$
$$+\frac{1.3.5.......(2n-5)n(n-1)\,(n-2)\,(n-3)x^{n-4}}{2.4.n!}.......$$
$$=\frac{1.3.5.....(2n-1)}{n!}\left[x^n-\frac{n(n-1)\,x^{n-2}}{2(2n-1)!}+\frac{n(n-1)\,(n-2)(n-3)}{2.4(2n-1)\,(2n-3)}x^{n-4}....\right]$$

$$= \frac{1.2.3.4\ldots(2n-1)(2n)}{n!\,2.4.6\ldots(2n)}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2}\ \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}\ldots..\right]$$

$$= \frac{(2n)!}{n!\ 2^n(1.2.3\ldots..n)}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}\ \ldots.\right\}$$

$$= \frac{(2n)!}{2^n(n!)^2}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}\ \ldots.\right\}$$

$= P_n(x)$

$$\therefore \frac{1}{\sqrt{1-2xt+t^2}} = P_n(x)t^n$$

Problem

Prove that $P_n(1) = 1$, $P_n(-1) = (-1)^n$

Proof

We've $\frac{1}{\sqrt{1-2x+t^2}} = \sum P_n(x)t^n$

ie) $\sum P_n(x)t^n = \left(1-2xt+t^2\right)^{-1/2}$

Put $x = 1$

$$\sum P_n(x)t^n = \left(1-2t+t^2\right)^{-1/2}$$

$$= \left[(1-t)^2\right]^{-1/2}$$

$$\sum P_n(x)t^n = (1-t)^{-1}$$

$$= 1 + t + t^2 + t^3 + \ldots. + t^n + \ldots\ldots$$

Equating the coeff of t^n, weget

$$P_n(1) = 1$$

Put $x = -1$

$$\sum P_n(x)t^n = \left(1-2(-1)t+t^2\right)^{-1/2}$$

$$= (1 + 2t + t^2)^{-1/2}$$

$$= \left[(1+t)^2\right]^{-1/2}$$

$$\sum P_n(x)t^n = (1-2(-1)t+t^2)^{-1/2}$$

$$= 1 - t - t^2 - t^3 + \ldots (-1)^n t^n + \ldots$$

Equating coeff of t^n weget

$$P_n(-1) = (-1)^n.$$

Note :

$P_n(-1) = (-1)^n$ and $P_n(1) = 1$

$\therefore P_n(-1) = 1(-1)^n$

$= P_n(1)(-1)^n$

$\therefore P_n(-1) = (-1)^n P_n(1)$

Prove that $P_n(-x) = (-1)^n P_n(x)$

Soln

We've $\sum P_n(x)t^n = (1-2xt+t^2)^{-1/2}$

Put $x = -x$

$$\sum P_n(x)t^n = (1-2(x)t+t^2)^{-1/2}$$

$$= (1 + 2xt + t^2)^{-1/2}$$

$$= \left[1-2x(-t)+(-t)^2\right]^{-1/2}$$

$$= \sum P_n(x)(-t)^n$$

$$= \sum P_n(x)(-1)^n t^n$$

$$\therefore \sum P_n(x)t^n = (-1)^n \sum P_n(x)t^n$$

$$\therefore P_n(-x) = (-1)^n P_n(x)$$

Using Rodrigues formula prove that $P_0(x) = 1$, $P_1(x) = 1$, $P_2(x) = ½ (3x^2 -1)$ $P_3(x) = ½ (5x^3 - 3x)$

Solution:

The Rodrigues formula is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

n = 0

$$P_0(x) = \frac{1}{2^0 .0!} \frac{d^0}{dx^0}\left(x^2 - 1\right)^0$$

$$= \frac{1}{1} = 1$$

n = 1

$$P_1(x) = \frac{1}{2^1 .1!} \frac{d}{dx}\left(x^2 - 1\right)$$

$$= \frac{1}{2} \,.\, 2x$$

$$= x$$

n = 2

$$P_2(x) = \frac{1}{2^2 .2!} \frac{d^2}{dx^2}\left(x^2 - 1\right)^2$$

$$= \frac{1}{2^2 .2} \frac{d}{dx}\left(\frac{d}{dx}\left(x^2 - 1\right)^2\right)$$

$$= \frac{1}{2^2 .2} \frac{d}{dx}\left(2\left(x^2 - 1\right) .\, 2x\right)$$

$$= \frac{1}{2} \frac{d}{dx}\left(x^3 - x\right)$$

$$= \frac{1}{2}\left(3x^2 - 1\right)$$

n = 3

$$P_3(x) = \frac{1}{2^3 .3!} \frac{d^3}{dx^3}\left(x^2 - 1\right)^3$$

$$= \frac{1}{2^3 .3!} \frac{d^2}{dx^2}\left(\frac{d}{dx}\left(x^2-1\right)^3\right)$$

$$= \frac{1}{2^3 .3!} \frac{d^2}{dx^2}\left(3\left(x^2-1\right)^2 . 2x\right)$$

$$= \frac{1}{2^3} \frac{d}{dx}\left(\frac{d}{dx}\left(x^2-1\right)^2 . x\right)$$

$$= \frac{1}{2^3} \frac{d}{dx}\left(\frac{d}{dx}\left(x^4+1-2x^2\right)x\right)$$

$$= \frac{1}{2^3} \frac{d}{dx}\left(\frac{d}{dx}\left(x^5+x-2x^3\right)\right)$$

$$= \frac{1}{2^3} \frac{d}{dx}\left(5x^4+1-6x^2\right)$$

$$= \frac{1}{2^3}\left(20x^3-12x\right)$$

$$= \frac{1}{2^3} . 4\left(5x^3-3x\right)$$

$$= \frac{1}{2}\left(5x^3-3x\right)$$

Prove that $P_{2n+1}(0) = 0$ and $P_{2n}(0) = \frac{(-1)^n\ 1.3.....(2n-1)}{2^n\ . n!}$

Proof:

We know,

$$\sum P_n(x)t^n = \left(1-2xt+t^2\right)^{-1/2}$$

Put $x = 0$

$$\sum P_n(0)t^n = \left(1+t^2\right)^{-1/2}$$

$$= 1-1/2t^2 + \frac{1/2 . 3/2}{1.2}\left(t^2\right)^2 - \frac{1/2 . 3/2 . 5/2\left(t^2\right)^3}{1.2.3} +....+$$

$$\frac{(-1)^n 1/2 \,.\, 3/2 \,.\, 5/2 \ldots\left(\frac{2n-1}{2}\right)(t^2)^n}{1.2.3\ldots.n} + \ldots.$$

We find $P_n(0)$ is the coeff of t^n in the expansion of $(1 + t^2)^{-1/2}$. This expansion, contains only even powers of t, so the coeff of odd powers.

(ie) t^{2n+1} is zero

$\therefore P_{2n+1}(0) = 0$

Equating the coeff of t^{2n}

$$P_{2n}(0) = \frac{(-1)^n 1/2.3/2.5/2\ldots.\left(\frac{2n-1}{2}\right)}{1.2\ldots\ldots n}$$

$$P_{2n}(0) = \frac{(-1)^n 1.3.5\ldots(2n-1)}{2^n \,.\, n!}$$

Prove that the occurrence relation. $(n + 1) P_{n+1}(x) = (2n + 1)\, x\, P_n(x) - nP_{n-1}(x)$

Proof:

We know

$$\sum P_n(x)t^n = \left(1-2xt+t^2\right)^{-1/2} \qquad \ldots\ldots\ldots (1)$$

Diff w.r.t 't'

$$\sum nP_n(x)t^{n-1} = \frac{-1}{2}\left(1-2xt+t^2\right)^{-3/2} \left(-2x+2t\right)$$

$$= \frac{-1}{2}\left(1-2xt+t^2\right)^{-3/2} (-2)(x-t)$$

$$\sum nP_n(x)t^{n-1} = \left(1-2xt+t^2\right)^{-3/2} (x-t)$$

Multiplying both sides $(1 - 2xt + t^2)$

$$\sum nP_n(x)t^{n-1}\left(1-2xt+t^2\right) = \left(1-2x+t^2\right)^{-1/2}(x-t)$$

$$nP_n(x)t^{n-1} - 2xnP_n(x)t^n + nP_n(x)t^{n+1} = \sum P_n(x)t^n(x-t)$$

$$(n+1)P_n(x)t^n - 2xnP_n(x)t^n + (n-1)P_{n-1}(x)t^n = xP_n(x)t^n - P_n(x)t^{n+1}$$

$$= xP_n(x)t^n - P_{n-1}(x)t^n$$

$$\{(n+1)P_{n+1}(x) - 2x_nP_n(x) + (n-1)P_{n-1}(x) - xP_n(x) + P_{n-1}(x)\}t^n = 0$$

Equating the coeff of t^n to zero

$$(n+1)P_{n+1}(x) - 2x_nP_n(x) + (n-1)P_{n-1}(x) - xP_n(x) + P_{n-1}(x) = 0$$

$$(n+1)P_{n+1}(x) - xP_n(x)[2n+1] + P_{n-1}(x)[n-1+1] = 0$$

$$(n+1)P_{n+1}(x) - xP_n(x)(2n+1) + nP_{n-1}(x) = 0$$

$(n + 1) P_{n+1}(x) = (2n + 1) xP_n(x) - nP_{n-1}(x)$.

Hence Proved

Orthogonal Property of Legendre Polynomial

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & if \quad m \neq n \\ \dfrac{2}{2n+1} & if \quad m = n \end{cases}$$

Proof:

Let f(x) be any f_n with atleast n continuous derivatives on the interval $-1 \leq x \leq 1$

Consider the integral

$$I = \int_{-1}^{1} f(x)P_n(x)dx \qquad \text{(use Rodrigues form)}$$

$$= \int_{-1}^{1} f(x)\frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n dx$$

$$= \frac{1}{2^n.n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}(x^2-1)^n dx$$

$$= \frac{1}{2^n.n!}\left[f(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1}$$

$$-\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right) f'(x) dx$$

The expression in bracket vanishes at both the limits.

$$\therefore \left[f(x) \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n \right]_{-1}^{1} = 0$$

$$I = -\frac{1}{2^n \ . n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n dx$$

$$= (-1)^2 \ . \frac{1}{2^n n!} \int_{-1}^{1} f''(x) \frac{d^{n-2}}{dx^{n-2}}\left(x^2-1\right)^n dx$$

$$= (-1)^3 \ . \frac{1}{2^n . n!} \int_{-1}^{1} f''(x) \frac{d^{n-3}}{dx^{n-3}}\left(x^2-1\right)^n dx$$

$$= (-1)^n \frac{1}{2^n \ . n!} \int_{-1}^{1} f^{(n)}(x) \frac{d^{n-n}}{dx^{n-n}}\left(x^2-1\right)^n dx \qquad \text{….. (1)}$$

If $f_n(x) = P_m(x)$ with m < n

then $f^{(n)}(x) = 0$

$\therefore$ I = 0

$$\int_{-1}^{1} P_m(x) P_n(x) dx = 0 \text{ if } m \neq n$$

Second part:

Put $f(x) = P_n(x)$

We've $P_n(x) = \frac{(2n)!}{2^n . (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \ldots \right]$

$$P_n^{(n)}(x) = \frac{(2n)!}{2^n . (n!)^2} n!$$

$$P_n^{(n)}(x) = \frac{(2n)!}{2^n.n!}$$

(1) ⇒ $$I = (-1)^n . \frac{1}{2^n.n!} \int_{-1}^{1} P_n^{(n)}(x)\,(x^2-1)^n\,dx$$

$$= (-1)^n \frac{1}{2^n.n!} \frac{(2n)!}{2^n.n!} \int_{-1}^{1} (x^2-1)^n\,dx$$

$$= \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (1-x^2)^n\,dx$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}\, 2\int_{0}^{1} (1-x^2)^n\,dx \qquad \text{……..….. (2)}$$

If we change the variable by putting $x = \sin\theta$

$x = 0 \Rightarrow \theta = 0$ $\quad dx = \cos\theta\,.\,d\theta$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\int_0^1 (1-x^2)^n\,dx = \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)^n \cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\cos^2\theta)^n \cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta\;.\,d\theta$$

$$\left[\int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta = \frac{n-1}{n}\,.\,\frac{n-3}{n-2}\,.\,\frac{n-5}{n-4}\ldots\frac{2}{3}\,.\,1 \quad \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta = \frac{n-1}{n}\,.\,\frac{n-3}{n-2}\ldots\frac{1}{2}\,.\,\frac{\pi}{2}\right]$$

$$\therefore \int_0^1 (1-x^2)^n\,dx = \frac{2n}{2n+1}\,.\,\frac{2n-2}{2n-1}.\frac{2n-4}{2n-3}\ldots\frac{2}{3}.1$$

$$= \frac{2.4.6\ldots(2n-4)(2n-2)(2n)}{1.3.5\ldots.(2n-3)(2n-1)(2n+1)}$$

$$= \frac{2^n .n!\ 2.4.6...(2n-2)(2n)}{1.2.3.4.5.6....(2n-3)(2n-2)(2n-1)(2n)(2n+1)}$$

$$\therefore \int_0^1 (1-x^2)^n dx = \frac{2^n .n!\ 2^n\ n!}{(2n)!\ (2n+1)} = \frac{2^{2n} (n!)^2}{(2n)!\ (2n+1)}$$

Equation (2) ⇒

$$I = \frac{(2n)!}{2^{2n} (n!)^2}\ 2 \cdot \frac{2^{2n} (n!)^2}{(2n)!\ (2n+1)}$$

$$I = \frac{2}{2n+1} \text{ if m = n}$$

Problem

Prove that any function can be expressed as a series of Legendre Polynomial:

Proof:

Let f(x) be any function defined in $-1 \le x \le 1$

Let f(x) = $\Sigma a_n P_n(x)$

f(x) $P_m(x) = \Sigma a_n P_n(x) P_m(x)$

$$\int_{-1}^{1} f(x) P_m(x) dx = \sum a_n \int_{-1}^{1} P_n(x) P_m(x) dx$$

$$= a_0 \int_{-1}^{1} P_0(x) P_m(x) dx + a_1 \int_{-1}^{1} P_1(x) P_m(x) dx + ... + a_n \int_{-1}^{1} P_n(x) P_{n+1}(x) dx + ...$$

$$= a_0(0) + a_1(0) + ... + a_n \frac{2}{2n+1} + a_{n+1}(0) + ...$$

$$\int_{-1}^{1} f(x) P_m(x) dx = a_n \frac{2}{2n+1}$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x) dx$$

Giving different values for n, we get all the coefficient a_0, a_1, a_2, a_3, …..

Hence any function can be expressed as a series of Legendre Polynomial.

Problem:

Find first three terms of the Legendre's series of a) $f(x)=\begin{cases}0 & if \quad -1\le x\le 0\\ x & if \quad 0\le x\le 1\end{cases}$

b) f(x) = e^x c) $f(x)=\begin{cases}0 & if \quad -1\le x\le 0\\ 1 & if \quad 0\le x\le 1\end{cases}$

Proof

c) We've, $f(x)=\sum_{n=0}^{\infty} a_n P_n(x)$

Where $$a_n=(n+1/2)\int_{-1}^{1} f(x)P_n(x)dx$$

$$a_n=(n+1/2)\left\{\int_{-1}^{1} f(x)P_n(x)dx+\int_{0}^{1} f(x)P_n(x)dx\right\}$$

$$= (n+1/2)\left\{\int_{0}^{1} 0.P_n(x)dx+\int_{0}^{1} 1.P_n(x)dx\right\}$$

$$\therefore a_n=\left(n+\frac{1}{2}\right)\int_{0}^{1} P_n(x)dx$$

Put n = 0

$$a_0=1/2\int_{0}^{1} P_0(x)dx$$

$$= 1/2\int_{0}^{1} 1.dx$$

$$= 1/2\left[x\right]_0^1$$

$$= \frac{1}{2}(1-0)$$

$$= 1/2$$

Put n = 1

$$a_1=(1+1/2)\int_{0}^{1} P_1(x)dx$$

$$= 3/2\int_0^1 x\,dx$$

$$= 3/2\left[\frac{x^2}{2}\right]_0^1 \qquad = \frac{3}{2}.\frac{1}{2} \qquad = \frac{3}{4}$$

Put n = 2

$$a_2 = (2+1/2)\int_0^1 P_2(x)\,dx$$

$$= 5/2\int_0^1 \frac{1}{2}(3x^2-1)dx$$

$$= \frac{5}{4}\left[\frac{3x^3}{3}-x\right]_0^1$$

$$= \frac{5}{4}[1-1-0]$$

$$= \frac{5}{4}(0)$$

$$= 0$$

$$\therefore f(x) = a_0P_0(x)+a_1P_1(x)+a_2P_2(x)$$

$$= 1/2P_0(x)+3/4P_1(x)+0.P_2(x)$$

$$= 1/2.1+\frac{3}{4}.x+0$$

$$f(x) = \frac{3}{4}x+\frac{1}{2}$$

Orthogonal function

If a sequence of functions $\varphi_1(x)$, $\varphi_2(x)$ ….., $\varphi_n(x)$, …. defined on the interval $a \le x \le b$ has the properly that $\int_a^b \varphi_m(x)\varphi_n(x)dx = \begin{cases} 0 & if \quad m \ne n \\ \alpha_n \ne 0 & if \quad m = n \end{cases}$ then the φ_n are said to be Orthogonal functions on this interval.

Prove that any function f(x) can be expanded as a series of Orthogonal functions.

Proof:

Consider the Orthogonal function $\{\varphi_n(x)\}_{n=1,2,.....}$ defined on the interval a ≤x≤ b

$$\int_a^b \varphi_m(x)\varphi_n(x)dx = \begin{cases} 0 & if \quad m \neq n \\ \alpha_n \neq 0 & if \quad m = n \end{cases}$$

Let f(x) be a function defined in a ≤x≤ b

$$\therefore f(x) = \sum_{n=1}^{\infty} a_n\varphi_n(x)$$

$$f(x)\varphi_n(x) = \sum_{n=1}^{\infty} a_n\varphi_n(x)\,\varphi_n(x)$$

$$\int_a^b f(x)\varphi_n(x) = \sum_{n=1}^{\infty} a_n \int_a^b \varphi_n(x)\,.\,\varphi_n(x)dx + a_2\int_a^b \varphi_n(x)\varphi_2(x)dx$$

$$= a_1\int_a^b \varphi_n(x)\,.\,\varphi_n(x)dx + a_2\int_a^b \varphi_n(x)\varphi_2(x)dx + ... + a_n\int_a^b \varphi_n(x).\varphi_n(x)dx +$$

$$a_{n+!}\int_a^b \varphi_n(x).\,\varphi_{n+1}(x)dx + ...$$

$$= a_1(0) + a_2(0) + ... + a_n\alpha_n + a_{n+1}(0) + ...$$

$$\int_a^b f(x)\varphi_n(x)dx = a_n\alpha_n$$

$$\therefore a_n = \frac{1}{\alpha_n}\int_a^b f(x)\,\varphi_n(x)dx$$

Hence, we get the series for f(x) interms of Orthogonal function

Least squares approximation:

Let f(x) be a function defined on the interval -1 ≤x≤ 1

$$\therefore f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \text{ where } a_n = \frac{2}{2n+1}$$

Let f(x) be approximated as a polynomial of degree n.

Let the polynomial be P(x)

∴ We can take $P(x) = b_0P_0(x) + b_1P_1(x) + b_2P_2(x) + \dots + b_nP_n(x)$

we claim that $a_k = b_k$, k = 0, 1, 2, ..., n so that P(x) is a Legendre series

Let $I = \int_{-1}^{1} [f(x) - P(x)]^2 dx$

By principle of Least square this integral must be minimum

$$\therefore I = \int_{-1}^{1} \left[f(x) - \sum_{k=0}^{n} b_k P_k(x) \right]^2 dx$$

$$= \int_{-1}^{1} \left[f(x)^2 - 2 f(x) \sum b_k P_k(x) + \sum b_k^2 P_k^2(x) \right] dx$$

$$= \int_{-1}^{1} [f(x)]^2 dx - 2 \sum b_k \int_{-1}^{1} f(x) P_k(x) dx + \sum b_k^2 \int_{-1}^{1} P_k^2(x) dx$$

$$= \int_{-1}^{1} [f(x)]^2 dx - 2 \sum b_k \cdot \frac{2a_k}{2k+1} + \sum b_k^2 \cdot \frac{2}{2k+1}$$

$$= \int_{-1}^{1} [f(x)]^2 dx + \frac{2}{2k+!} \left\{ \sum b_k^2 - 2 \sum a_k b_k \right\}$$

$$= \int_{-1}^{1} (f(x))^2 dx + \frac{2}{2k+1} \left\{ \sum a_k^2 + \sum b_k^2 - 2 \sum a_k \sum b_k - \sum a_k^2 \right\}$$

$$I = \int_{-1}^{1} [f(x)]^2 dx + \frac{2}{2k+1} \sum (a_k - b_k)^2 - \frac{2}{2k+1} \sum a_k^2$$

we observe that I is minimum.

When, $\Sigma(a_k - b_k)^2 = 0 \ \forall k$.

$\Rightarrow a_k = b_k \forall$ k.

Unit III

Bessel's Functions

The differential equation $x^2y'' + xy' +(x^2 - p^2)\, y = 0$ where p is a constant is called Bessel's equation and its solution are known as Bessel's functions.

Given equation is

$$x^2y''+ xy' + (x^2 - p^2)\, y = 0 \qquad \ldots.. (1)$$

$$\Rightarrow \quad y'' + \frac{1}{x}y' + \left(\frac{x^2 - y^2}{x^2}\right)y = 0$$

$$p(x) = \frac{1}{x} \text{ and } Q(x) = \frac{x^2 - p^2}{x^2}$$

Here p(x) and Q(x) are not analytic at the put $x = 0$

$$\therefore xP(x) = 1, \qquad x^2Q(x) = x^2 - p^2$$

$$p_0 = 1, \qquad q_0 = - p^2$$

p(x), x^2Q(x) are analytic at the put $x = 0$

The indicial eqn is

$$m(m - 1) + m\, p_0 + q_0 = 0$$

$$m^2 - m + m - p^2 = 0$$

$$m^2 - p^2 = 0$$

$$\Rightarrow m^2 = p^2$$

$$\Rightarrow m = \pm p.$$

$m_1 = p$, $m_2 = -p$

The Frobenius series solution is

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

$$y = x^p \sum_{n=0}^{\infty} a_n x^n$$

$$y = \sum a_n x^{n+p}$$
$$y' = \sum (n+p) a_n x^{n+p-1}$$

$$y'' = \sum (n+p)\,(n+p-1)\; a_n x^{n+p-2}$$

Sub in (1)

$$x^2 \sum (n+p)(n+p-1) a_n\; x^{n+p-2} + x\sum (n+p) a_n x^{n+p-1} + \left(x^2 - p^2\right) \sum a_n x^{n+p} = 0$$

$(n + P)\ (n + p -1)a_n x^{n+p} + (n + p)\ a_n x^{n+p} + a_n x^{n+p+2} - p^2 a_n x^{n+p} = 0$

$\Rightarrow (n + p)\ (n + p - 1)\ a_n x^{n+p} + (n + p)\ a_n x^{n+p} + a_{n-2} x^{n+p} - p^2\ a_n x^{n+p} = 0$

$[(n + p)\ (n + p - 1)\ a^n + (n + p)\ a_n + a_{n-2} - p^2\ a_n]\ x^{n+p} = 0$

Equating the coeff of x^{n+p} to zero

$(n + p)\ (n + p -1)a_n + (n + p)a_n + a_{n-2} - p^2 a_n = 0$

$[(n + p)\ (n + p - 1) + (n + p) - p^2]\ a_n = - a_{n-2}$

$[(n +p)\ [n + p -1 +1] - p^2]\ a_n = -a_{n-2}$

$[(n + p)\ (n + p) - p^2]\ a_n = -a_{n-2}$

$((n + p)^2 - p^2)\ a_n = -a_{n-2}$

$$\therefore a_n = \frac{-a_{n-2}}{(n+p)^2 - p^2}$$

$$a_n = \frac{-a_{n-2}}{n^2 + p^2 + 2np - p^2}$$

$$a_n = \frac{-a_{n-2}}{n^2 + 2np}$$

$$a_n = \frac{-a_{n-2}}{n(n+2p)}$$

Put n = 1

$$a_1 = \frac{-a_{-1}}{1(1+2p)}$$

Since the assumed series does not contains negative powers, so $a_{-1} = 0$.

$\therefore a_1 = 0$

Put n = 2

$$a_2 = \frac{-a_0}{2(2+2p)}$$

Put n = 3

$$a_3 = \frac{-a_1}{3(2p+3)} = 0$$

$$a_3 = 0$$

Put n = 4

$$a_4 = \frac{-a_2}{4(2p+4)}$$

$$= \frac{a_0}{2.4(2p+2)(2p+4)}$$

Put n = 5

$$a_5 = \frac{-a_3}{5(2p+5)} = 0$$

Put n = 6

$$a_6 = \frac{-a_4}{6(2p+6)}$$

$$= \frac{-a_0}{2.4.6\,(2p+2)(2p+4)\,(2p+6)}$$

$\therefore$ The solution is

$$y = x^m \sum_{n=0}^{\infty} a_n x^n$$

$$= x^p(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$= x^p\left(a_0 + 0 - \frac{a_0}{2(2p+2)}x^2 + 0x^3 + \frac{a_0}{2.4(2p+2)(2p+4)}\right)$$
$$+ 0x^5 - \frac{a_0}{2.4.6(2p+2)(2p+4)(2p+6)} + \dots.$$

$$= x^p\left[a_0 - \frac{a_0x^2}{2(2P+2)} + \frac{a_0x^4}{2.4(2p+2)(2p+4)}\right.$$
$$\left. - \frac{a_0x^6}{2.4.6(2p+2)(2p+4)(2p+6)} + \dots\dots\right]$$

$$= x^p\left[a_0 - \frac{a_0x^2}{2(2P+2)} + \frac{a_0x^4}{2.4(2p+2)(2p+4)} - \frac{a_0x^6}{2.4.6(2p+2)(2p+4)(2p+6)} + \dots\dots\right].$$

$$y = x^p a_0\left[1 - \frac{x^2}{2(2P+2)} + \frac{x^4}{2.4(2p+2)(2p+4)} - \frac{x^6}{2.4.6(2p+2)(2p+4)(2p+6)} + \dots\dots\right].$$

$$= x^p a_0\left[1 - \frac{x^2}{1!2^2(p+1)} + \frac{x^4}{2!2^4(p+1)(p+2)} - \frac{x^6}{3!\,2^6(p+1)(p+2)(p+3)} + \dots\dots\right].$$

Take $a_0 = \dfrac{1}{2^p\,.\,p!}$

$$\therefore y = x^p\,\frac{1}{2^p\,.\,p!}\left[1 - \frac{x^2}{1!2^2(p+1)} + \frac{x^4}{2!2^4(p+1)(p+2)} - \frac{x^6}{3!\,2^6(p+1)(p+2)(p+3)} + \dots\dots\right].$$

$$= x^p\,\frac{1}{2^p\,.\,p!}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!\;2^{2n}\,(p+1)(p+2)\dots.(p+n)}$$

$$= \frac{1}{p!}\left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{2n}}{n!\;(p+1)(p+2)\dots.(p+n)}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{2n+p}}{p!\,n!\;(p+1)(p+2)\dots.(p+n)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\ (n+p)!}$$

This is called the Bessel's functions of the first kind of order p, and it is denoted by $J_p(x)$

$$\therefore J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\ (n+p)!}$$

To find the solution corresponding to m = -p

We've $y = x^{-p} \sum_{n=0}^{\infty} a_n x^n$

Proceeding as above, we get

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{n!\ (n-p)!}$$

This is called the Bessel's function of the second kind of order p, and it is denoted by $J_{-p}(x)$

∴ The complete solution is

$$y = C_1 J_p(x) + C_2 J_{-p}(x)$$

The Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ has two independent solutions $J_p(x)$ & $J_{-p}(x)$ only when p is not an integer

Particular cases:

put p = 0

∴ Bessel's equation is $x^2y'' + xy' + x^2y = 0$

The solution is

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\ n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{(n!)^2}$$

Put p = 1

∴ Bessel's equation is $x^2y'' + xy' + (x^2 - 1)y = 0$

$m = \pm 1$

$m_1 = 1, \quad m_2 = -1$

$m_1 - m_2 = 1 + 1 = 2$ is a +ive integer

∴ The equation has only on frobenius series solution (corresponding to $m_1 = 1$).

ie) $J_1(x)$ is the only Frobenius series solution. But $J_{-1}(x)$ cannot be taken as the solution of the differential equation. Further we can prove $J_{-1}(x)$ is indepent of $J_1(x)$.

In this case we find the second solution using $y_2 = \upsilon y_1$

$$\upsilon = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$$

$$= \int \frac{1}{J_1^2(x)} e^{-\int 1/x\, dx} dx$$

$$= \int \frac{1}{J_1^2(x)} e^{-\log x} dx$$

$$= \int \frac{1}{J_1^2(x)} e^{\log \frac{1}{x}} dx$$

$$= \int \frac{1}{J_1^2(x)} \frac{1}{x} dx$$

$$\upsilon = \int \frac{1}{xJ_1^2(x)} dx$$

$$y_2 = j_1(x) \int \frac{1}{xJ_1^2(x)} dx$$

This is denoted by Y_1

∴The complete solution is $y = C_1J_1(x) + C_2\, Y_1(x)$

In general consider the Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$.

The roots of the indicial equation are $m_1 = p$, $m_2 = -p$

the corresponding solution are $J_p(x)$ & $J_{-p}(x)$

If p is a +ive integer then $m_1 - m_2 = p + p = 2p$

$\therefore \exists$ only one Frobenius series solution and it corresponds to $m_1 = p$

ie) The Frobenius series solution is $J_p(x)$. Here we cannot take $J_{-p}(x)$ as the other solution

In this case we take $y_2 = \upsilon y_1$

$$\therefore y_2 = J_p(x) \int \frac{1}{x J_p^{\,2}(x)}\, dx$$

This is denoted by Y_p

$\therefore$ The complete solution is $y = C_1 J_p(x) + C_2\, Y_p(x)$

Problem

Find the first three terms of the Legendre's series of

a) $f(x) = \begin{cases} 0 & if \quad -1 \le x \le 0 \\ x & if \quad 0 \le x \le 1 \end{cases}$

b) $f(x) = e^x$

Proof:

i) $f(x) = \sum_{n=0}^{\infty} a_n p_n(x)$ where

$$a_n = (n + 1/2) \int_{-1}^{1} f(x) p_n(x)\, dx$$

$$a_n = (n + 1/2) \left[\int_{-1}^{0} f(x) p_n(x) dx + \int_{0}^{1} f(x) p_n(x) dx \right]$$

$$= (n + 1/2) \left[\int_{-1}^{0} 0\, p_n(x) dx + \int_{0}^{1} x p_n(x) dx \right]$$

$$a_n = (n + 1/2) \int_{0}^{1} x p_n(x) dx$$

Put n = 0

$$a_0 = (0 + 1/2) \int_{0}^{1} x p_0(x) dx$$

$$= 1/2 \int_{0}^{1} x \,.\, 1 dx$$

$$= 1/2 \left[\frac{x^2}{2} \right]_0^1$$

$$a_0 = \frac{1}{4}$$

Put n = 1

$$a_1 = (1+1/2)\int_0^1 x p_2(x) dx$$

$$a_1 = \frac{3}{2}\int_0^1 x \cdot x \; dx$$

$$= \frac{3}{2}\int_0^1 x^2 dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{2} \cdot \frac{1}{3}$$

$$a_1 = \frac{1}{2}$$

Put n = 2

$$a_2 = (2+1/2)\int_0^1 x p_2(x)\, dx$$

$$= 5/2 \int_0^1 x \cdot \frac{1}{2}(3x^2-1) dx$$

$$= \frac{5}{4}\int_0^1 (3x^3 - x)\, dx$$

$$= \frac{5}{4}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{5}{4}\left[\frac{3}{4} - \frac{1}{2}\right]$$

$$= \frac{5}{4}\left[\frac{3-2}{4}\right]$$

$$= \frac{5}{16}$$

$$f(x) = a_0\, p_0(x) + a_1 p_1(x) + a_2 p_2(x)$$

$$= \frac{1}{4} p_0(x) + \frac{1}{2}\, p_1(x) + \frac{5}{16} p_2(x)$$

$= \frac{1}{4}.1 + \frac{1}{2}.x + \frac{5}{16}.\frac{1}{2}(3x^2 - 1)$

$= \frac{1}{4} + \frac{x}{2} + \frac{15}{32}x^2 - \frac{5}{32}$

$f(x) = \frac{15}{32}x^2 + \frac{x}{2} + \frac{3}{32}$

ii) $f(x) = e^x$

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ where

$a_n = (n + 1/2)\int_{-1}^{1} f(x) p_n(x)\, dx$

$a_n = (n + 1/2)\int_{-1}^{1} e^x p_n(x) dx$

Put n = 0

$a_0 = (0 + 1/2)\int_{-1}^{1} e^x p_0(x)\, dx$

$= 1/2 \int_{-1}^{1} e^x .1\, dx$

$= \frac{1}{2}\int_{-1}^{1} e^x .1\, dx$

$= \frac{1}{2}\left[e^x\right]_{-1}^{1}$

$= \frac{1}{2}\left[e^1 - e^{-1}\right]$

Put n = 1

$a_1 = (1 + 1/2)\int_{-1}^{1} e^x p_1(x) dx$

$= \frac{3}{2}\int_{-1}^{1} e^x . x\, dx$

$= \frac{3}{2}\left[\left[xe^x\right]_{-1}^{1} - \int_{-1}^{1} e^x\, dx\right]$

$= \frac{3}{2}\left[e^1 + e^{-1} - \left[e^x\right]_{-1}^{1}\right]$

$$= \frac{3}{2}\left(e^{1}+e^{-1}-e^{1}+e^{-1}\right)$$

$$= \frac{3}{2}\,.\,2e^{-1}$$

$$a_1 = 3e^{-1}$$

Put n = 2

$$a_2 = (2+1/2)\int_{-1}^{1} e^{x} p_2(x)\,dx$$

$$= \frac{5}{2}\int_{-1}^{1} e^{x}\,.\,\frac{1}{2}\left(3x^{2}-1\right)dx$$

$$= \frac{5}{4}\int_{-1}^{1}\left(3e^{x}x^{2}-e^{x}\right)dx$$

$$= \frac{5}{4}\left\{\int_{-1}^{1} 3x^{2}e^{x}dx - \int_{-1}^{1} e^{x}dx\right.$$

$$= \frac{5}{4}\left\{\left[3x^{2}e^{x}\right]_{-1}^{1} - \int_{-1}^{1} 6xe^{x}dx - \left[e^{x}\right]_{-1}^{1}\right\}$$

$$= \frac{5}{4}\left\{3e^{1}-3e^{-1} - 6\int_{-1}^{1} xe^{x}dx - e^{1}+e^{-1}\right\}$$

$$= \frac{5}{4}\left\{3e^{1}-3e^{-1} - 6\left\{\left[xe^{x}\right]_{-1}^{1} - \int_{-1}^{1} e^{x}dx\right\} - e^{1}+e^{-1}\right\}$$

$$= \frac{5}{4}\left\{3e^{1}-3e^{-1} - 6\left[\left(e^{1}+e^{-1}\right)-\left[e^{x}\right]_{-1}^{1}\right]-e^{1}+e^{-1}\right\}$$

$$= \frac{5}{4}\left\{3e^{1}-3e^{-1} - 6e^{1} - 6e^{-1} + 6\left(e^{1}-e^{-1}\right)-e^{1}+e^{-1}\right\}$$

$$= \frac{5}{4}\left\{3e^{1}-3e^{-1} - 6e^{1} - 6e^{-1} + 6e^{1} - 6e^{-1} - e^{1}+e^{-1}\right\}$$

$$= \frac{5}{4}\left\{2e^{1} - 14e^{-1}\right\}$$

$$= \frac{5}{2}e^{1} - \frac{35e^{-1}}{2}$$

$$= \frac{1}{2}\left(5e^{1} - 35e^{-1}\right)$$

$$f(x) = a_0p_0(x) + a_1p_1(x) + a_2p_2(x)$$

$$= \frac{1}{2}\left(e^{1} - e^{-1}\right)p_0(x) + 3e^{-1}p_1(x) + \frac{1}{2}\left(5e^{1} - 35e^{-1}\right)p_2(x)$$

$$= \frac{1}{2}\left(e^{1} - e^{-1}\right) + 3xe^{-1} + \frac{1}{2} \cdot \frac{1}{2}\left(5e^{1} - 35e^{-1}\right)\left(3x^{2} - 1\right)$$

Gamma function:

Gamma function is defined by $\lceil p = \int_0^\infty e^{-t} t^{p-1}$dt, (p > 0)

Now, t.p

$$\Gamma p + 1 = p \lceil p$$

$$\Gamma 1 = 1$$

$$\Gamma p + 1 = p!$$

Proof:

$$\Gamma p = \int_0^\infty e^{-t} \ t^{p+1}\, dt$$

$$\Gamma p + 1 = \int_0^\infty e^{-t} \ t^{p+1-1}\, dt$$

$$= \int_0^\infty e^{-t} \ t^{p}\, dt \qquad u = t^p, \ du = pt^{p-1}\, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-t} t^{p}\, dt \qquad \int dv = \int e^{-t} \quad v = -e^{-t}$$

$$= \lim_{b\to\infty}\left\{\left[-t^{p}e^{-t}\right]_0^b + \int_0^b e^{-t}\,p.t^{p-1}dt\right\}$$

$$= \lim_{b\to\infty}\left\{0 + p\int_0^b e^{-t}\ .\ t^{p-1}dt\right\}$$

$$= \lim_{b\to\infty} p\int_0^b e^{-t}t^{p-1}dt$$

$$= p\int_0^b e^{-t}t^{p-1}dt$$

$$= p\Gamma p$$

$$\therefore \Gamma p + 1 = p\,\Gamma p$$

T.p $\Gamma 1 = 1$

w.k.t $\Gamma p = \int_0^\infty e^{-t}\ t^{p-1}\,dt$

$$\Gamma 1 = \int_0^\infty e^{-t}\ t^{1-1}\,dt$$

$$= \int_0^\infty e^{-t}\ dt$$

$$= -\left[e^{-t}\right]_0^\infty$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= -\left[0-1\right]$$

$$= 1$$

$$\therefore \Gamma 1 = 1$$

T.p $\Gamma p + 1 = p!$

w.k.t $\Gamma p + 1 = p\lceil p$

$$= p(p-1)\lceil(p-1)$$

$= \quad p(p-1)(p-2)\Gamma p-2$

$\vdots$

$= \quad p(p-1)(p-2)\ldots 3.2.1$

$= \quad p!$

$\therefore \Gamma p+1 \quad = \quad p!$

Problem

Prove that $\Gamma\, {}^{1}/_{2} = \sqrt{\pi}$

Proof:

w.k.t $\quad \Gamma p \quad = \quad \int_0^\infty e^{-t} t^{p-1} dt$

Put p = ½

$$\Gamma\, {}^{1}/_{2} = \int_0^\infty e^{-t} t^{1/2-1} dt$$

$$= \int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt$$

put $t = s^2$, $dt = 2sds$

$$\Gamma\, {}^{1}/_{2} = \int_0^\infty \frac{e^{-s^2}}{\sqrt{s^2}} 2s\, ds$$

$$= 2\int_0^\infty \frac{e^{-s^2}}{s} s\, ds$$

$$= 2\int_0^\infty e^{-s^2} ds$$

We've $\Gamma \frac{1}{2} = 2\int_0^\infty e^{-x^2} dx$

Also $\Gamma \frac{1}{2} = 2\int_0^\infty e^{-y^2} dy$

$\Gamma \frac{1}{2} \Gamma \frac{1}{2} = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$

Put $x = r\cos\theta \qquad y = r\sin\theta$

$\frac{\partial x}{\partial r} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$x^2 + y^2 = (r^2 \cos^2\theta + r^2 \sin^2\theta)$

$= r^2$

$dx\, dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\ d\theta$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\, d\theta$

$= (r\cos^2\theta + r\sin^2\theta)\, dr\, d\theta$

$= r(\cos^2\theta + \sin^2\theta)\, dr\, d\theta$

$= r\, dr\, .\, d\theta$

$\therefore dxdy = r\, dr\, d\theta$

$0 \le r \le \infty, \ 0 \le \theta \le \pi/2$

$\therefore (\Gamma \frac{1}{2})^2 = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2}\, r dr\, d\theta$

$= 4\int_0^\infty \left[e^{-r^2}\ r\, dr\, \theta \right]_0^{\pi/2}$

$$= \quad 4\pi/2\int_0^\infty e^{-r^2}\, r\, dr$$

$$= \quad 2\pi\int_0^\infty e^{-z} dz$$

$$\left(\Gamma\, {}^1\!/_2\right)^2 \quad = \quad \pi\int_0^\infty e^{-z} dz$$

$$= \quad \pi\left[-e^{-z}\right]_0^\infty$$

$$= \quad -\pi\left(e^{-\infty} - e^0\right)$$

$$= \quad -\pi(0-1)$$

$$= \quad \pi$$

$$\therefore \left(\Gamma\, {}^1\!/_2\right)^2 \quad = \quad \pi$$

$$\Gamma\, {}^1\!/_2 \quad = \quad \sqrt{\pi}$$

Prove that i) $\frac{d}{dx} J_0(x) = -J_1(x)$

ii) $\frac{d}{dx}\left(xJ_1(x)\right) = xJ_0(x)$

Proof

We know,

$$J_p(x) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(n+p)!}$$

$$J_0(x) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{n!\,n!}$$

$$= \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$J_0(x) = 1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+....$$

$$J_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n+1}}{n!(n+1)!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}n!(n+1)!}$$

$$= \frac{x}{2}-\frac{x^3}{2^3 .1!\ 2!}+\frac{x^5}{2^5\ 2!\ 3!}+.....$$

$$\frac{d}{dx}(J_0(x)) = \frac{d}{dx}\left(1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+...\right)$$

$$= \frac{-2x}{2^2(1!)^2}+\frac{4x^3}{2^4(2!)^2}-\frac{6x^5}{2^6(3!)^2}+...$$

$$= \frac{-x}{2}+\frac{x^3}{2^4}-\frac{x^5}{2^6 .3!}+....$$

$$= -\left(\frac{x}{2}-\frac{x^3}{2^3 .2!}+\frac{x^5}{2^5.2!3!}....\right)$$

$$= -J_1(x)$$

$$xJ_1(x) = \frac{x^2}{2}-\frac{x^4}{2^3 1!\ 2!}+\frac{x^6}{2^5.2!3!}$$

$$\frac{d}{dx}(xJ_1(x)) = \frac{d}{dx}\left(\frac{x^2}{2}-\frac{x^4}{2^3 1!\ 2!}+\frac{x^6}{2^5 2!\,3!}-.....\right)$$

$$= \left(\frac{2x}{2}-\frac{4x^3}{2^3 .1!2!}+\frac{6x^5}{2^5 2!3!}\right)$$

$$= \left(x-\frac{x^3}{(2!)^2}+\frac{x^5}{2^5 2!}\right)$$

$$\frac{d}{dx}(xJ_1(x)) = x\left(1-\frac{x^2}{(2!)^2(1!)^2}+\frac{x^4}{2^4(2!)^2}\right)$$

$= \quad xJ_0(x)$

Properties of Bessel's function

Prove that $\frac{d}{dx}\left(x^p J_p(x)\right) = x^p J_{p-1}(x)$

Solution

We know,

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!\,(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n=p}\, n!\,(n+p)!}$$

$$\therefore x^p J_p(x) = \frac{\sum_{n=0`}^{\infty} (-1)^n x^{2n+2p}}{2^{2n+p}\, n!\,(n+p)!}$$

$$\frac{d}{dx} J\left(x^p J_p(x)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2p) x^{2n+2p-1}}{2^{2n+p}\, n!\,(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\ 2(n+p) x^{2n+2p-1}}{2^{2n+p}\, n!\,(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+p) x^{2n+2p-1}}{2^{2n+p-1}\, n!\,(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p-1} (n+p)}{2^{2n+p-1}\, n!\,(n+p)(n+p-1)!}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p-1}}{2^{2n+p-1}\, n!\,(n+p-1)!}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p-1}}{n!\,(n+p-1)!}$$

$$= x^p J_{p-1}(x)$$

$\therefore \frac{d}{dx} J\left(x^p J_p(x)\right) \quad = \quad x^p J_{p-1}(x)$

Prove that $J_p{}'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$

Proof:

w.k.t

$\frac{d}{dx}\left(x^p J_p(x)\right) \quad = \quad x^p J_{p-1}(x)$

$x^p J_p{}'(x) + p x^{p-1} J_p(x) = \quad x^p J_{p-1}(x)$

Dividing through by x^p

$J_p{}'(x) + \frac{p}{x} J_p(x) \quad = \quad J_{p-1}(x)$

Prove that $J_p{}'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$

Proof:

We've

$\frac{d}{dx}\left(x^{-p} J_p(x)\right) \quad = \quad -x^{-p} J_{p+1}(x)$

$x^{-p} J_p{}'(x) + J_p(x)(-p)x^{-p-1} = -x^{-p} J_{p+1}(x)$

Dividing through by x^{-p}

$J_p{}'(x) - \frac{p}{x} J_p(x) \quad = \quad -J_{p+1}(x)$

Prove that $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$ and $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

We know

$$j_p{}'(x) + \frac{p}{x} J_p(x) \quad = \quad J_{p-1}(x) \qquad \text{….. (1)}$$

$$j_p{}'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x) \quad \text{..... (2)}$$

(1) + (2)

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$$

(1) – (2)

$$\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$$

Note:

The above formula helps to find any Bessel function interms of other Bessel's function

Solution

We've $$\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$$

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$$

Put p = 1

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

Put p = 2

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$= \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$= \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

$$J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

Put p = 3

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left[\left(\frac{8}{x^2}-1\right)J_1(x) - \frac{4}{x}J_0(x)\right] - \left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

$$= \left(\frac{48}{x^3} - \frac{6}{x}\right)J_1(x) - \frac{24}{x^2}J_0(x) - \frac{2}{x}J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right)J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right)J_0(x)$$

Prove that $\int x^p J_{p-1}(x)dx = x^p J_p(x) + c$ and $\int x^{-p} J_{p+1}(x)dx = -x^{-p} J_p(x) + c$

Proof

We've

$$\frac{d}{dx}\left(x^p J_p(x)\right) = x^p J_{p-1}(x)$$

ie) $$x^p J_{p-1}(x)dx = \frac{d}{dx}\left(x^p J_p(x)\right)$$

$\int ing$

$$x^p J_{p-1}(x)dx = x^p J_p(x) + c$$

Also, we've

$$\frac{d}{dx}\left(x^{-p} J_p(x)\right) = -x^{-p} J_{p+1}(x)$$

$$x^{-p} J_{p+1}(x) = \frac{-d}{dx}\left(x^{-p} J_p(x)\right)$$

$\int ing$

$$\int x^{-p} J_{p+1}(x)dx = -x^{-p} J_p(x) + c$$

Prove that when p is a positive integer $J_{-p}(x) = (-1)^p J_p(x)$

Proof

We know that

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n=P}}{n!(n+p)!}$$

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{n!\,(n-p)!}$$

For n = 0, 1, 2,……p-1

(n – 1)! is ±∞

$$\therefore J_{-p}(x) = \sum_{n=p}^{\infty} \frac{(-1)^n (x/2)^{2n-p}}{n!\,(n-p)!}$$

$$J_{-p}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+p} (x/2)^{2(m+p)-p}}{(m+p)!\,m!} \qquad \text{(put } n - p = m\text{)}$$

$$= (-1)^p \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+p}}{m!\,(m+p)!}$$

$$= (-1)^p J_p(x)$$

$$\therefore J_{-p}(x) = (-1)^p J_p(x)$$

Note :

From the above we observe that $J_p(x)$ and $J_{-p}(x)$ are not linearly independent and so the solution of the Bessel's equation when p is an integer cannot be taken in the form $y = C_1 J_p(x) + C_2 J_{-p}(x)$.

In this case we can take $J_p(x)$ as one solution and the other solution

$$Y_p(x) = J_p(x) \int \frac{1}{x J_p^{\,2}(x)}\, dx$$

∴ The complete solution is $y = C_1 J_p(x) + C_2 Y_p(x)$.

Assuming $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\,\sin x$ and $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\,\cos x$. Find $J_{3/2}(x)$, $J_{5/2}(x)$, $J_{-3/2}(x)$, $J_{-5/2}(x)$

Solution

We know

$$\frac{2p}{x}J_p(x) = J_{p-1}(x)+J_{p+1}(x)$$

$$J_{p+1}(x) = \frac{2p}{x}J_p(x)-J_{p-1}(x)$$

Put p = 1/2

$$J_{3/2}(x) = \frac{2\times 1/2}{x}\,J_{1/2}(x)-J_{-1/2}(x)$$

$$= \frac{1}{x}J_{1/2}(x)-J_{-1/2}(x)$$

$$J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x-\sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$$

Put p = 3/2

$$J_{5/2}(x) = \frac{2\times 3/2}{x}\,J_{3/2}(x)-J_{1/2}(x)$$

$$= \frac{3}{x}J_{3/2}(x)-J_{1/2}(x)$$

$$= \frac{3}{x}\left[\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)\right]-\sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x^2}-\frac{3\cos x}{x}-\sin x\right]$$

$$J_{p-1}(x) = \frac{2p}{x}J_p(x)-J_{p+1}(x)$$

Put p = -1/2

$$J_{3/2}(x) = \frac{2\times -1/2}{x} J_{-1/2}(x) - J_{-1/2+1}(x)$$

$$= \frac{-1}{x} J_{-1/2}(x) - J_{1/2}(x)$$

$$= \frac{-1}{x}\sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{-\cos x}{x} - \sin x\right)$$

$$= -\sqrt{\frac{2}{\pi x}}(\cos x + \sin x)$$

Put p = -3/2

$$J_{-5/2}(x) = \frac{2\times(-3/2)}{x} J_{-3/2}(x) - J_{-3/2+1}(x)$$

$$= \frac{-3}{x} J_{-3/2}(x) - J_{-1/2}(x)$$

$$= \frac{-3}{x}\left[-\sqrt{\frac{2}{\pi x}}(\cos x + \sin x)\right] - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \frac{3}{x}\sqrt{\frac{2}{\pi x}}(\cos x + \sin x) - \sqrt{\frac{2}{\pi x} \cos x}$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\cos x + \frac{3}{x}\sin x - \cos x\right]$$

$$= \sqrt{\frac{2}{\pi x}\left[\left(\frac{3}{x} - 1\right)\cos x + \frac{3}{x}\sin x\right]}$$

Proceeding like this we get $J_{7/2}(x)$, $J_{-7/2}(x)$, $J_{9/2}(x)$, $J_{-9/2}(x)$,………

In general we can find $J_{m+1/2}(x)$ all this functions are combinations of elementary functions sin$x$ and cos x and so for all integrals value of m, the Bessel's function $J_{m+1/2}$ are elementary function

Problem

Prove that $\frac{d}{dx}\left(x^{-p} J_p(x)\right) = -x^{-p} J_{p+1}(x)$

Solution

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n+p} n!\,(n+p)!}$$

X^{ly} by x^{-p}

$$x^{-p}J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+p} x^{-p}}{2^{2n+p} n!\,(n+p)!}$$

$$x^{-p}J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+p} n!\,(n+p)!}$$

$$\frac{d}{dx}\left(x^{-p} J_p(x)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n\, x^{2n-1}}{2^{2n+p} n!\,(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n x^{2n-1}}{2^{2n+p-1} n!\,(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n+p-1} (n-1)!\,(n+p)!}$$

x^{ly}&÷ by -1

$$= -\sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2^{2n+p-1} (n-1)!\,(n+p)!}$$

x^{ly}&÷ by x^{p}

$$= -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n+p-1}}{2^{2n+p-1} (n-1)!\,(n+p)!}$$

$$= -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n+p-1-1+1}}{2^{2n+p-1-1+1} (n-1)!\,(n+p-1+1)!}$$

$$= -x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n-2+p+1}}{2^{2n-2+p+1} (n-1)!\,(n+p-1+1)!}$$

$$= \quad -x^{-p}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}x^{2(n-1)+p+1}}{2^{2n-2+p+1}(n-1)!\,(n+p-1+1)!}$$

$$= \quad -x^{-p}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}(x/2)^{2(n-1)+p+1}}{2^{2n-2+p+1}(n-1)!\,((n-1)+(p+1))!}$$

Put n – 1 = m

$$\frac{d}{dx}\left(x^{-p}J_p(x)\right) = -x^{-p}\sum_{n=0}^{\infty}\frac{(-1)^m(x/2)^{2(m)+p+1}}{2^{2n-2+p+1}m!\,(m+(p+1))!}$$

$$= \quad -x^{-p}J_{p+1}(x)$$

Zero's of Bessel function

Consider the Bessel function $J_p(x)$ for $p \geq 0$. We know $J_p(x)$ is an infinite series. So $J_p(x)$ has infinite number of five zero's to the equation $J_p(x) = 0$ has infinite no of +ive roots. For any given $p \geq 0$ we take the +ive roots as $\lambda_1, \lambda_2, \lambda_3, \ldots$

Clearly $J_p(\lambda_n) = 0 \ \forall \ n \geq 0$

Orthogonal property of the Bessel function

Prove that $\int_0^1 xJ_p(\lambda mx)J_p(\lambda_n x)dx = 0$ if m ≠n and $\int_0^1 xJ_p{}^2(\lambda_n x)dx = \frac{1}{2}J_{p+1}{}^2(\lambda_n)$ if m = n

Proof:

Consider the Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad \ldots\ldots (1)$$

$$\Rightarrow \quad y'' + \frac{1}{x}y' + \frac{(x^2 - p^2)}{x^2}y = 0$$

For this equation $y = J_p(x)$ is the solution

put $u(x) = y(ax)$

$u' = y'a$

$u'' = y''a^2$

$\therefore$ sub $y' = \frac{u'}{a}$, $y'' = \frac{u''}{a^2}$ and (ax) for x in equation (1)

$$(ax)^2 \frac{u''}{a^2} + ax\frac{u'}{a} + \left(a^2x^2 - p^2\right)u \quad = \quad 0$$

$$\Rightarrow \quad x^2u'' = xu' + \left(a^2x^2 - p^2\right)u \quad = \quad 0$$

$$\Rightarrow \quad u'' + \frac{1}{x}u' + \left(a^2 - \frac{p^2}{x^2}\right)u \quad = \quad 0 \qquad \text{..... (2)}$$

Since $J_p(x)$ in the solution of the given equation we get $J_p(x)$ is the solution of equation (2) lll^{rly} $J_p(bx)$ will the solution of

$$v'' + \frac{1}{x}v' + \left(b^2 - \frac{p^2}{x^2}\right)v \quad = \quad 0 \qquad \text{..... (3)}$$

(2) × v - (3) × u

$$\Rightarrow (u''v - v''u) + \frac{1}{x}(u'v - v'u) + \left(a^2 - b^2\right)uv \quad = \quad 0$$

$$\Rightarrow x(u''v - v''u) + (u'v - v'u) + \left(a^2 - b^2\right)xuv \quad = \quad 0 \qquad \text{..... (4)}$$

We've

$$\frac{d}{dx}[(u'v - v'u)x] \quad = \quad (u'v - v'u)1 + x[u''v + u'v' - v''u - u'v']$$

$$= \quad (u'v - v'u) + x(u''v - v''u)$$

$$(4) \Rightarrow \quad \frac{d}{dx}[(u'v - v'u)x] + \left(a^2 - b^2\right)xuv \quad = \quad 0$$

Integrate between 0 and 1

$$[(u'v - v'u)x]_0^1 + \left(a^2 - b^2\right)\int_0^1 xuv\ dx \quad = \quad 0$$

$$[u'(1)v(1) - v'(1)u(1)] - 0 + \left(a^2 - b^2\right)\int_0^1 xuvdx \quad = \quad 0 \qquad \text{..... (5)}$$

We've,

$$u(x) \quad = \quad J_p(ax) \quad = \quad J_p(\lambda_m x)$$

$$v(x) \quad = \quad J_p(bx) \quad = \quad J_p(\lambda_n x)$$

$u(1)$	=	$J_p(\lambda_m)$	=	0	[$\therefore$ a and b are distinct positive zeros of λ_m and λ_n of $J_p(x)$]
$v(1)$	=	$J_p(\lambda_n)$	=	0	

$\therefore$ (5) $\Rightarrow$ $$0+\left(a^2-b^2\right)\int_0^1 xuv\,dx = 0$$

$$\left(\lambda_m{}^2-\lambda_n{}^2\right)\int_0^1 xJ_p(\lambda_m x)J_p(\lambda_m x)J_p(\lambda_n x)dx = 0$$

If m ≠ n, the positive zeros λ_m and λ_n are distinct

$\therefore \lambda_m{}^2 - \lambda_n{}^2 \neq 0$

$$\therefore \int_0^1 xJ_p(\lambda_m x)J_p(\lambda_n x)dx = 0 \text{ if } m \neq n$$

ii) If m = n

We've

$$u''+\frac{u'}{x}+\left(a^2-\frac{p^2}{x^2}\right)u = 0$$

Multiply by $2x^2$ u'

$$2x^2u''u'+2xu'^2+2a^2x^2u'u-2p^2u'u = 0$$

$$\frac{d}{dx}\left(x^2u'^2\right)+2a^2x^2u'u+2a^2u^2x-2a^2u^2x-2p^2u'u = 0$$

$$\frac{d}{dx}\left(x^2u'^2\right)+\frac{d}{dx}\left(a^2x^2u^2\right)-\frac{d}{dx}\left(p^2u^2\right) = 2a^2u^2x$$

$$\frac{d}{dx}\left(x^2u'^2+a^2x^2u^2-p^2u^2\right) = 2a^2u^2x$$

Integrate between 0 and 1

$$\left[x^2u'^2+a^2x^2u^2-p^2u^2\right]_0^1 = 2a^2\int_0^1 u^2x\,dx$$

$$\left[x^2u'^2+\left(a^2x^2-p^2\right)u^2\right]_0^1 = 2a^2\int_0^1 u^2x\,dx$$

$$u'(1)^2 + (a^2 - p^2)u^2(1) = 2a^2\int_0^1 u^2 x\,dx$$

$$u(x) = J_p(\lambda_n x)$$

$$u(1) = J_p(\lambda_n) = 0$$

$$u'(x) = J_p'(\lambda_n x)\lambda_n$$

$$u'(1) = J_p'(\lambda_n)\,\lambda_n$$

$$u_1'^2(1) = J_p'^2(\lambda_n)(\lambda_n)^2$$

$$\therefore 2a^2\int_0^1 u^2 x dx = J_p'^2\lambda_n^2$$

$$2\lambda_n^2\int_0^1 u^2 x dx = J_p'^2(\lambda_n)(\lambda_n)^2$$

$$\int_0^1 u^2 x dx = \frac{1}{2}J_p'^2(\lambda_n)$$

$$\int_0^1 xJ_p^2(\lambda_n x)dx = \frac{1}{2}J_p'^2(\lambda_n) \quad \text{..... (6)}$$

We know,

$$J_p'(x) - \frac{p}{x}J_p(x) = -J_{p+1}(x)$$

Put $x = \lambda_n$

$$J_p'(\lambda_n) - \frac{p}{\lambda_n}J_p(\lambda_n) = -J_{p+1}(\lambda_n)$$

$$J_p'(\lambda_n) - 0 = -J_{p+1}(\lambda_n)$$

$$\therefore J_p'(\lambda_n) = -J_{p+1}(\lambda_n)$$

$$J_p'^2(\lambda_n) = J_{p+1}^2(\lambda_n)$$

$(6) \Rightarrow \int_0^1 x J_p^{\,2}(\lambda_n x)dx \quad = \quad \frac{1}{2} J_{p+1}^{\,2}(\lambda_n)$

Bessel's series

Let f(x) be a real value function defined in $0 \le x \le 1$. Let $\lambda_1, \lambda_2, \ldots$ be the +ive Zero's of the Bessel's function $J_p(x)$ for any $p \ge 0$ then we can write $f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$

$$f(x) = a_1 J_p(\lambda_1 x) + a_2 J_p(\lambda_2 x) + \ldots + a_n J_p(\lambda_n x) + a_{n+1} J_p(\lambda_{n+1} x) + \ldots$$

This series is called the Bessel's series of f(x)

To find the coefficient of the Bessel's series f(x)

The Bessel series for f(x) given by $f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$

$$\therefore f(x) = a_1 J_p(\lambda_1 x) + a_2 J_p(\lambda_2 x) + \ldots.. + a_n J_p(\lambda_n x) + a_{n+1} J_p(\lambda_{n+1} x) + \ldots. \qquad \ldots..(1)$$

Where $\lambda_1, \lambda_2, \ldots$, are the +ive zero's of $J_p(x)$ for $p \ge 0$.

we know,

$$\int_0^1 x J_p(\lambda_n x) J_p(\lambda_m x) dx \quad = \quad 0 \text{ if } m \ne n$$

and $\int_0^1 x J_p^{\,2}(\lambda_n x) dx \quad = \quad \frac{1}{2} J_{p+1}^{\,2}(\lambda_n) \text{ if } m = n$

(1) $\Rightarrow$ $x f(x) J_p(\lambda_n x) = a_1 x J_p(\lambda_1 x) J_p(\lambda_n x) + a_2 x J_p(\lambda_2 x) J_p(\lambda_n x) + \ldots.. + a_n x J_p^{\,2}(\lambda_n x) + a_{n+1} x J_p(\lambda_n x) J_p(\lambda_{n+1} x) + \ldots.$

$$\int_0^1 x f(x) J_p(\lambda_n x) dx = a_1 \int_0^1 x J_p(\lambda_1 x) J_p(\lambda_n x) dx + a_2 \int_0^1 x J_p(\lambda_2 x) J_p(\lambda_n x) dx + \ldots.. + a_n \int_0^1 x J_p^{\,2}(\lambda_n x) dx + a_{n+1} \int_0^1 x J_p(\lambda_n x) J_p(\lambda_{n+1} x) dx + \ldots.$$

$$= a_1 \times 0 + a_2 \times 0 + \ldots + a_n \frac{1}{2} J_{p+1}^{\,2}(\lambda_n) + a_{n+1} \times 0 + \ldots$$

$$= \frac{a_n}{2} J_{p+1}^{\ 2}(\lambda_n)$$

$$\therefore a_n = \frac{2}{J_{p+1}^{\ 2}(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x)\, dx$$

Compute the Bessel series of function f(x) = 1 for the interval $0 \leq x \leq 1$ interms of the function $J_0(\lambda_n x)$ where the λ_n's are the +ive zero's of $J_0(x)$.

Proof

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$$

Where $a_n = \frac{2}{J_{p+1}^{\ 2}(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x)\, dx$

Here f(x) = 1, p = 0

$$a_n = \frac{2}{J_1^{\ 2}(\lambda_n)} \int_0^1 x \,.\, 1 \,.\, J_0(\lambda_n x)\, dx$$

$$= \frac{2}{J_1^{\ 2}(\lambda_n)} \int_0^1 x \,.\, J_0(\lambda_n x)\, dx$$

$$= \frac{2}{J_1^{\ 2}(\lambda_n)} \left[\frac{x \,.\, J_1(\lambda_n x)}{\lambda_n} \right]_0^1$$

$$= \frac{2}{J_1^{\ 2}(\lambda_n)} \left[\frac{J_1(\lambda_n)}{\lambda_n} - 0 \right]$$

$$= \frac{2}{J_1^{\ 2}(\lambda_n)} \cdot \frac{J_1(\lambda_n)}{\lambda_n}$$

$$= \frac{2}{\lambda_n J_1(\lambda_n)}$$

The series is

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_1(\lambda_n)} J_0(\lambda_n x)$$

Problem

If f(x) is defined by $f(x) = \begin{cases} 1 & if \quad 0 \le x \le 1/2 \\ 1/2 & if \quad x = 1/2 \\ 0 & if \quad 1/2 < x \le 1 \end{cases}$ then show that $f(x) = \sum_{n=1}^{\infty} \frac{J_1(\lambda_n/2)}{\lambda_n J_1^2(\lambda_n)} J_0(\lambda_n x)$ where the λ_n's are the +ive zero's of $J_0(x)$

Solution:

We've $f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$

Where $a_n = \frac{2}{J_{p+1}^2(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x) dx$

Here p = 0

$$a_n = \frac{2}{J_{p+1}^2(\lambda_n)} \left\{ \int_0^{1/2} x f(x) J_p(\lambda_n x) dx + \int_{1/2}^{1/2} x f(x) J_p(\lambda_n x) dx + \int_{1/2}^{1} x f(x) J_p(\lambda_n x) dx \right.$$

$$= \frac{2}{J_1^2(\lambda_n)} \left\{ \int_0^{1/2} x . 1 . J_0(\lambda_n x) dx + \int_{1/2}^{1/2} x . \frac{1}{2} . J_0(\lambda_n x) dx + \int_{1/2}^{1} x . 0 . J_0(\lambda_n x) dx \right.$$

$$= \frac{2}{J_1^2(\lambda_n)} \int_0^{1/2} x . J_0(\lambda_n x) dx$$

$$= \frac{2}{J_1^2(\lambda_n)} \left[\frac{x J_1(\lambda_n x) dx}{\lambda_n} \right]_0^{1/2}$$

$$= \frac{2}{J_1^2(\lambda_n)} \left(\frac{1/2 \; J_1(\lambda_n/2)}{\lambda_n} - 0 \right)$$

$$= \frac{2}{\lambda_n J_1^2(\lambda_n)} \cdot \frac{1}{2} J_1(\lambda_n/2)$$

$$= \frac{J_1(\lambda_n / 2)}{\lambda_n J_1^{\ 2}(\lambda_n)}$$

$$\therefore \text{f}(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$$

$$= \sum_{n=1}^{\infty} \frac{J_1(\lambda_{n/2})}{\lambda_n J_1^2(\lambda_n)} J_0(\lambda_n x)$$

If f(x) = x^p for the interval $0 \le x \le 1$, show that its Bessel series in the functions $J_p(\lambda_n x)$, where the λ_n are the positive zeros of $J_p(x)$, is $x^p = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{p+1}(\lambda_n)} J_p(\lambda_n x)$

Proof:

We've $\quad$ $f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$

Where $\quad$ $a_n = \frac{2}{J_{p+1}^{\ 2}(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x) dx$

$$a_n = \frac{2}{J_{p+1}^{\ 2}(\lambda_n)} \int_0^1 x \, . \, x^p \ J_p(\lambda_n x) dx$$

$$= \frac{2}{J_{p+1}^{\ 2}(\lambda_n)} \int_0^1 x^{p+1} \ J_p(\lambda_n x) dx$$

$$= \frac{2}{J_{p+1}^{\ 2}(\lambda_n)} \left[\frac{x^{p+1} \ J_{p+1}(\lambda_n x)}{\lambda_n} \right]_0^1$$

$$= \frac{2}{J_{p+1}^{\ 2}(\lambda_n)} \left[\frac{J_{p+1}(\lambda_n x)}{\lambda_n} - 0 \right]$$

$$= \frac{2}{\lambda_n \ J_{p+1}(\lambda_n)}$$

$$\therefore \text{f}(x) = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{p+1}(\lambda_n)} J_p(\lambda_n x)$$

$$x^p = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{p+1}(\lambda_n)} J_p(\lambda_n x)$$

Boundary value problem and methods of successive approximation

Consider the differential equation of the first order y' = f(x,y)with the initial condition when $x=x_0$, y = y_0.

The problem of finding a solution to the diff equation satisfying the initial condition [boundary values] is called a Boundary value Problem [B.V.P].

Now, consider B.V.P.

y' = f(x,y), y(x_0) = y_0 **.........** (1)

The solution of this equation is not always possible, by the methods of solving first order diff equation. So the approximation solution is obtained by the method successive approximation.

We have,

y' = f(x,y)

Suppose f(x,y) is continuous in some interval containing x_0.

Integrating between x and x_0

$$[y(x)]_{x_0}^{x} = \int_{x_0}^{x} f(x,y).dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y).dx$$

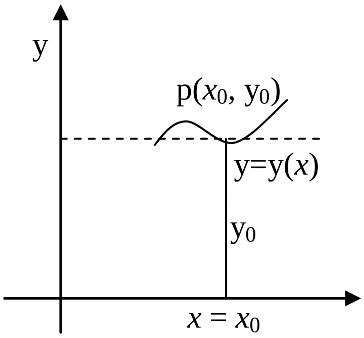
$$y(x) = y(x_0) + \int_{x_0}^{x} f(x,y).dx$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y).dx \qquad \text{......... (2)}$$

This integral equation is equivalent to the given equation with the boundary condition.

So the solution of the B.V.P (1) is same as the solution of the integral equation.
Now, for solving the integral equ (2) we apply methods of successive approximation.

Picard's Method



The approximate soln of B.V.P.

$y' = f(x,y)$, $y(x_0) = y_0$. is given by the integral equation.

$$y = y_0 + \int_{x_0}^{x} f(x,y).dx$$

The solutions must be a continuous curve y = y(x) passing through (x_0, y_0).

As a first approximation take $y = y_0$.

This is a straight line through (x_0, y_0) parallel to x-axis.

By successive approximation what we active is the improvement of this straight line into a curve which is very closed to the solution of the B.V.P.

The method is given below the integral equation is $y = y_0 + \int_{x_0}^{x} f(x,y).dx$ For convenience we use dummy variable 't' in the place of x, with in the integral equ

$$\therefore y = y_0 + \int_{x_0}^{x} f[t, y(t)].dt\ .$$

First approximation is $y_1 = y_0 + \int_{x_0}^{x} f(t, y_0(t))dt$.

Second approximation $y_2 = y_0 + \int_{x_0}^{x} f(t, y_1(t))dt$.

Third approximation $y_3 = y_0 + \int_{x_0}^{x} f(t, y_2(t))dt$. etc……. $y_n = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt$

Solve the B.V.P. y′ = y, y(0) = 1

Solution:

The corresponding integral equ.

$$y = y_0 + \int_{x_0}^{x} y(t).dt$$

$$y_0 = 1$$

$$y_1 = y_0 + \int_0^x y_0(t).dt$$

$$= 1 + \int_0^x 1.dt$$

$$= 1 + x.$$

$$y_2 = y_0 + \int_0^x y_1(t).dt$$

$$= 1 + \int_0^x (1+t).dt$$

$$= 1 + \left[t + \frac{t^2}{2} \right]_0^x$$

$$= 1 + x + \frac{x^2}{2}$$

$$y_3 = y_0 + \int_0^x y_2(t).dt$$

$$= 1 + \int_0^x 1 + t + \frac{t^2}{2} dt$$

$$y_3 = 1 + \left[t + \frac{t^2}{2} + \frac{t^3}{2.3} \right]_0^x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$\vdots$

$$y_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{n-1}}{(n-1)!}$$

Continuing this process infinite values of times, (i. e) takes n⟶∞,
We get,

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\therefore y = e^x$$

Note:

The solution of B.V.P. obtained by applying the methods of solving diff equ is called the exact solution.

By successive approximation we by to get a soln., which is approximate to the exact solution.

In some cases the solution obtained by successive approximation coincides with the exact solution.

For example:

Consider the above B.V.P., y' = y, y(0) = 1.

$y' = y$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{y} = dx.$$

$\int$ ing

$\log y = x + A.$

$x = 0, y = 1$

$\Rightarrow A = 0.$

$\therefore \log y = x$

$y = e^x$

Which is the exact solution, we find this is same as the solution, obtained by successive approximation.

2) Solve the B.V.P y' = x +y, y(0) = 1, by Picard's method, compare the solution with the exact value.

$$y = y_0 + \int_{x_0}^{x} f[t, y(t)].dt$$

$$y_0 = 1$$

$$y_1 = y_0 + \int_0^x [t + y_0(t)].dt$$

$$= \quad 1+\int_0^x (t+1)dt$$

$$= \quad 1+\frac{x^2}{2}+x$$

$$y_2 \quad = \quad y_0+\int_0^x [t+y_1(t)].dt$$

$$= \quad 1+\int_0^x \left[t+1+t+\frac{t^2}{2}\right]dt$$

$$= \quad 1+\left[\frac{t^2}{2}+t+\frac{t^2}{2}+\frac{t^3}{2.3}\right]_0^x$$

$$= \quad 1+x+x^2+\frac{x^3}{2.3}$$

$$y_3 \quad = \quad y_0+\int_0^x [t+y_2(t)].dt$$

$$= \quad 1+\int_0^x \left[t+1+t+t^2+\frac{t^3}{6}\right]dt$$

$$= \quad 1+\int_0^x \left(1+2t+t^2+\frac{t^3}{6}\right)dt$$

$$= \quad 1+\left(t+\frac{2t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4.6}\right)_0^x$$

$$= \quad 1+x+x^2+\frac{x_3}{3}+\frac{t^4}{4.6}$$

$$y_4 \quad = \quad y_0+\int_0^x [t+y_3(t)].dt$$

$$= \quad 1+\int_0^x \left[t+1+t+t^2+\frac{t^3}{3}+\frac{t^4}{24}\right]dt$$

$$= \quad 1+\left[t+\frac{2t^2}{2}+\frac{t^3}{3}+\frac{t^4}{12}+\frac{t^5}{24\times 5}\right]_0^x$$

$$= \quad 1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{120}$$

Taking n⟶∞

$$y_n \quad = \quad 1+x+x^2+\frac{x^3}{3}+\frac{x^4}{3.4}+\frac{x^5}{3.4.5}+\frac{x^6}{3.4.5.6}+.......$$

$$= \quad 1+x+2\left[\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}+.....\right]$$

$$= \quad 1+x+2\left[\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+.....\right]$$

$$= \quad 1+x+2\left(e^x-1-x\right)$$

y = $2e^x - x - 1$

To find exact solution:

y' = x+y, y(o) = 1.

$$\frac{dx}{dy}=x+y$$

$$\frac{dx}{dy}-y=x.$$

Hence P(x) = -1

Q(x) = x

Solution is

$$ye^{\int Pdx}=\int Qe^{\int Pdx}.dx+c$$

u = x, du = dx, dv = e^{-x}, v = $-e^{-x}$

$$ye^{-\int dx}=\int x.e^{-x}\,dx+c$$

$$ye^{-x}=\int x.e^{-x}\,dx+c$$

$$=-xe^{-x}-\int -e^{-x}1.dx+c$$

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

When $x = 0$, $y = 1$

$$1.\, e^{0} = 0-1+c.$$

$$c = 2$$

$$\therefore \quad ye^{-x} = -xe^{-x}-e^{-x}+2$$

$y = -x-1+2e^{x}$ is the exact solution.

Find the exact solution of initial value problem $y' = 2x(1+y)$, $y(0) = 0$ starting with $y_0(x) = 0$. Calculate $y_1(x)$, $y_2(x)$, $y_3(x)$ and compare with exact solution

Solution:

Given equation is

$$y' = 2x(1+y)$$

$$y(0) = 0$$

$$\therefore y_0 = 0$$

$$y_1 = y_0 + \int_0^x 2t(1+0).dt$$

$$= 0+\int_0^x 2t.dt$$

$$= x^2$$

$$y_2 = 0 + \int_0^x 2t(1+t^2).dt$$

$$= \int_0^x 2t.dt + \int_0^x 2t^3.dt.$$

$$= \left[\frac{2t^2}{2}\right]_0^x + \left[\frac{2t^4}{4}\right]_0^x$$

$$= x^2 + \frac{x^4}{2}$$

$$y_3 = 0+\int_0^x 2t\left(1+t^2+\frac{t^4}{2}\right).dt$$

$$= \int_0^x \left(2t+2t^3+\frac{2t^5}{2}\right).dt$$

$$= \left[\frac{2t^2}{2}+\frac{2t^4}{4}+\frac{t^6}{6}\right]_0^x$$

$$= x^2+\frac{x^4}{2}+\frac{x^6}{6}$$

Proceeding like this

$$y = x^2+\frac{x^4}{2}+\frac{x^6}{6}+\ldots$$

$$1+y = 1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\ldots$$

$$1+y = e^{x^2}$$

To find the exact value

$$\frac{dx}{dy} = 2x(1+y)$$

$$\frac{dy}{1+y} = 2x.dx$$

$$\log(1+y) = \frac{2x^2}{2}+c$$

Initially, we have $y = 0$, $x = 0$.

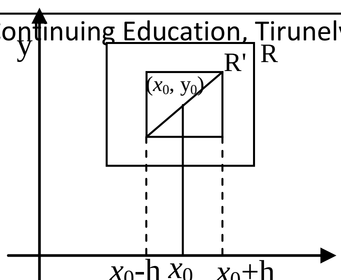
$\log 1 = 0+c \quad \Rightarrow c = 0.$

$\log(1+y) = x^2$

$1+y = e^{x^2}$

Picard's Theorem

Let f(x, y) and $\frac{\partial f}{\partial y}$ be continuous for of x and y in a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is any interior point of R_1 then f a number h>0, with in the property that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has one and only one solution $y = y(x)$ on the interval $|x-x_0| \le h$.



Proof:

We know that, every solution of the initial value problem. x

$$y' = f(x, y),\ y(x_0) = y_0 \qquad \text{......... (1)}$$

is also a continuous solution of the integrate equation.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)).dt \qquad \text{......... (2)}$$

So that equ (1) has a unique soln on the interval $|x-x_0|\leq h$ iff (2) has a unique continuous solution on the same interval.

By successive approximation, we get a sequence of function $y_n(x)$ defined by

$$\left.\begin{aligned} y_o(x) &= y_0 \\ y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0(t)).dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t)).dt \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t)).dt \end{aligned}\right\} \qquad \text{(A)}$$

and this sequence $\{y_n(x)\}$ converges to a solution of the integral equation (2)

Now, we can write

$$y_n(x) = y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \ldots\ldots + [y_n(x) - y_{n-1}(x)]$$

So, $y_n(x)$ is a partial sum of the series

$$y_n(x) = y_0(x) + \sum_{n=1}^{\infty}[y_n(x) - y_{n-1}(x)] \qquad \text{......... (3)}$$

So convergence of the sequence (A) is equivalent to the convergence of the series (3)

Now we shall find out the positive number h>0 which defines on the interval $|x-x_0|\leq h$ and we S.T.

i. The series (3) converges to a function $y(x)$
ii. $y(x)$ is a continuous solution of (2)
iii. $y(x)$ is the only continuous solution of (2)

We have assumed that f(x, y) and $\frac{\partial f}{\partial y}$ are continuous function on the rectangle R.

But R is closed and bounded.

∴ F constants M and K
S.T $|f(x, y)| \leq M$ .... (4)

and $\left|\frac{\partial f}{\partial y} f(x, y)\right| \leq K$ .... (5)
for all points (x, y) in R.

Now, if (x, y_1) and (x, y_2) are district points in R, with the same x-coordinate. Then,

By Mean value theorem,

$$\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} = \frac{\partial}{\partial y}(f(x, y^*).$$

Where $y_1 < y^* < y_2$

$$\therefore \left|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| = \left|\frac{\partial}{\partial y} f(x, y^*)\right|$$

$$\therefore \left|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| \leq K \quad \text{(by (5))}$$

$$\therefore |f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \quad \text{......... (6)}$$

For any points (x, y_1) and (x, y_2) in R, that lie on the same vertical line.

Let us choose h to be any positive number such that Kh < 1 .... (7)

and the rectangle R' defined by the inequalities $|x-x_0| \leq h$ and $|y-y_0| \leq Mh$ is contained in R.

∴ (x_0, y_0) is an interior point of R.

Now T.P. (i) the series (3) converges to a function y(x)
The series (3) is

$$y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \ldots + [y_n(x) - y_{n-1}(x)] + \ldots$$

It is enough to prove.

$|y_0(x)| + |[y_1(x) - y_0(x)]| + |[y_2(x) - y_1(x)]| + \ldots + |[y_n(x) - y_{n-1}(x)]| + \ldots$ .... (8)

Converges

Let us estimate the term $|y_n(x)-y_{n-1}(x)|$. Each of the function $y_n(x)$ has a graph that lies in R' and hence in R.

Now $y_0(x) = y_0$ so the points $(t, y_0(t))$ are in R.

Equation (4) ⇒

$|f(t,y_0(t))| \leq M$ and

We have,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t)).dt$$

$$\therefore y_1(x) - y_0 = \int_{x_0}^{x} f(t, y_0(t)).dt$$

$$\therefore |y_1(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_0(t) dt\right|$$

$$\leq \int_{x_0}^{x} |f(t, y_0(t)| dt$$

$$= M \int_{x_0}^{x} |dt|$$

$$\therefore |y_1(x) - y_0| \leq M|x - x_0|$$

$$\therefore |y_1(x) - y_0| \leq M h$$

|||rly $|y_2(x) - y_0| \leq M h$

$$|y_3(x) - y_0| \leq M h$$

$$|y_n(x) - y_0| \leq M h$$

$y_1(x)$ is continuous
Since a continuous function on a closed interval has a maximum.

Define a constant ‘a’ by

a = max $|y_1(x)-y_0|$

and $|y_1(x)-y_0(x)| \leq a$

Now, the points $(t,y_1(t))$ and $(t,y_0(t))$ lie in R'

So, (6) ⇒

$$|f(t,y_1(t)) - f(t,y_0(t)| \leq K|y_1(t)-y_0(t)| \leq ka$$

Again from (A)

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t,y_0(t)).dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t,y_1(t)).dt$$

$$\therefore |y_2(x)-y_1(x)| = \left|\int_{x_0}^{x} f(t,y_1(t))\,dt - \int_{x_0}^{x} f(t,y_0(t))\,dt\right|$$

$$= \left|\int_{x_0}^{x}[f(t,y_1(t)-f(t,y_0(t))]dt\right|$$

$$\leq \left|\int_{x_0}^{x}|f(t,y_1(t)-f(t,y_0(t))|dt\right|$$

$$\leq \int_{x_0}^{x} Ka|dt|$$

$$= Ka\int_{x_0}^{x}|dt|$$

$$= Ka|x-x_0|$$

$$\therefore |y_2(x)-y_1(x)| \leq Kah$$

|||rly $|f(t,y_2(t)) - f(t,y_1(t)| \leq k|y_2(t)-y_1(t)| \leq K.Kah$

$= K^2$ ah.

So $\therefore |y_3(x) - y_2(x)| = \left| \int_{x_0}^{x} [f(t, y_2(t)) - f(t, y_1(t)].dt \right|$

$$\leq \int_{x_0}^{x} |f(t, y_2(t)) - f(t, y_1(t))|.dt$$

$$\leq K^2 ah \int_{x_0}^{x} |dt|$$

$$\leq K^2 ah |x - x_0|$$

$$= K^2 ah^2$$

$\therefore |y_3(x) - y_2(x)| \leq a(Kh)^2$

|||rly $|y_4(x) - y_3(x)| \leq a(Kh)^3$

$|y_5(x) - y_4(x)| \leq a(Kh)^4$

$|y_n(x) - y_{n-1}(x)| \leq a(Kh)^{n-1}$

etc

Now the series (8) is

$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| + \ldots\ldots + |y_n(x) - y_{n-1}(x)| + \ldots.$

$\leq |y_0(x)| + a + a(Kh) + a(Kh)^2 + \ldots. + a(Kh)^{n-1} + \ldots$

$\leq |y_0(x)| + \dfrac{a}{1 - Kh}$ [$\because$ Kh<1. the series is cgt]

The series (8) is convergent.

∴ The series (3) is converges to a sum y(x) and $y_n(x) \longrightarrow y(x)$.

(ii) To prove y(x) is a continuous solution of (2)

[The above argument shows not only that $y_n(x)$ converges to y(x) in the interval, but also this convergence is uniform. This means that by choosing n to be sufficiently large, we can make $y_n(x)$ as close as we please to y(x) for all x in the interval].

Given ∑=0 a positive integer no s.t $n \geq n_0$.

We have, $|y(x)-y_n(x)| < \sum$ for all x in the interval.

Since each $y_n(x)$ is clearly continuous.

The uniform convergence implies that the limit function $y(x)$ is also continuous.

T.P $y(x)$ is actually a solution of (2)

We must S.T. x

$$y(x) - y_0 - \int_{x_0}^{x} f(t,(t))\,dt = 0$$

We have, (2)

$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big).dt$$

$$\therefore y_1(x) - y_0 - \int_{x_0}^{x} f\big(t, y(t)\big).dt = 0 \qquad \text{......... (9)}$$

Also, we have,

$$y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big).dt$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big).dt$$

$$y(x) - y_n(x) = \int_{x_0}^{x} f\big(t, y(t)\big).dt - \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

$$y(x) - y_n(x) = \int_{x_0}^{x} \big[f(t, y(t)) - f(t, y_{n-1}(t))\big]dt$$

$$y(x) - y_n(x) - \int_{x_0}^{x} f\big(t, y(t)\big) - f(t, y_{n-1}(t))\,dt = 0 \qquad \text{......... (10)}$$

From (9) and (10)

$$y(x) - y_0 - \int_{x_0}^{x} f\big(t, y(t)\big)dt = y(x) - y_n(x) - \int_{x_0}^{x} f(t, y(t)) - f(t, y_{n-1}(t)).dt$$

$$\left| y(x) - y_0 - \int_{x_0}^{x} f\big(t, y(t)\big)dt \right| = \left| y(x) - y_n(x) + \int_{x_0}^{x} f(t, y_{n-1}(t)) - f(t, y(t)).dt \right|$$

$$\leq |y(x) - y_n(x)| + \left| \int_{x_0}^{x} f(t, y_{n-1}(t)) - f(t, y(t))dt \right|$$

$$\leq |y(x) - y_n(x)| + Kh.\max |y_{n-1}(x) - y(x)| \qquad \text{......... (11)}$$

The uniform convergence of $y_n(x)$ to $y(x)$ now implies that, the R.H.S (11) can be made as small as, we please,

By taking n, sufficiently large,

R.H.S of (11) $\longrightarrow 0$

$$\therefore \left| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t)\, dt \right| = 0$$

$$\therefore y(x) \quad = \quad y_0 + \int_{x_0}^{x} f\left(t, y(t)\right).dt$$

$\therefore$ $y(x)$ is a continuous soln of equ (2)

iii) T.P. $y(x)$ is the only continuous soln of (2)

Let us assume that $\bar{y}(x)$ is also a continuous solution of (2) on the interval $|x-x_0| \leq h$.

We shall prove that $\bar{y}(x) = y(x)$
We know that the graph of $\bar{y}(x)$ lies in R' and hence in R.

Let us suppose that the graph of $\bar{y}(x)$ leaves R'

$\Rightarrow$ F an x_1 such that

$|x_1-x_0| < h$.

$|\bar{y}(x_1)-y_0| = Mh$

Now, $|\bar{y}(x)-y_0| < Mh$ if $|x-x_0| < |x_1-x_0|$

$$\therefore \frac{|\bar{y}(x_1 - y_0|}{|x_1 - x_0|} = \frac{Mh}{|x_1 - x_0|} > \frac{Mh}{h} = M$$

$$\therefore \frac{|\bar{y}(x_1) - y_0|}{|x_1 - x_0|} > M \qquad \text{......... (12)}$$

By mean value theorem F a number x^* between x_0 and x_1 such that

$$\frac{|\bar{y}(x_1) - y_0|}{|x_1 - x_0|} = |\bar{y}(x^*)|$$

$$= \left| f(x^*, \bar{y}(x^*)) \right| \leq M$$

$$\therefore \frac{|\bar{y}(x_1) - y_0|}{|x_1 - x_0|} \leq M \qquad \text{......... (13)}$$

Since the pt (x^*,$\bar{y}(x^*)$) lies in R'

Equ (12) and (13) gives the contradiction.

Which shows that no point with the property of x_1 can exists, so the graph of $\bar{y}(x)$ lies in R'

Now $\bar{y}(x)$ and y(x) are both solutions of (2),

We write

$$|\bar{y}(x) - y(x)| = \left| \int_{x_0}^{x} f(t, \bar{y}(t) - f(t, y(t) dt \right|$$

Since the graphs of $\bar{y}(x)$ and y(x) both lie in R'

Equ (6) gives.

$$|\bar{y}(x) - y(x)| \leq Kh \max |\bar{y}(x) - y(x)|$$

So max $|\bar{y}(x) - y(x)| \leq Kh \max |\bar{y}(x) - y(x)|$

$$\Rightarrow \max |\bar{y}(x) - y(x)| = 0$$

$$\Rightarrow |\bar{y}(x) - y(x)| = 0$$

$\Rightarrow \bar{y}(x) = y(x)$ for every x in the interval $|x-x_0| \leq h$

∴ y(x) is the soln of (2)

Hence the proof.

Lipschitz condition

Let f(x, y) be any function define in a region R. If F a five number K s.t. $|f(x, y_1) - f(x, y_1)| \leq K |y_1 - y_2| \forall (x, y_1), (x, y_2)$ in the region, them f is said to satisfy Lipschitz condition. The number k is called Lipschitz constant.

Note:

Picard's theorem is also known as local existence and uniqueness theorem.

Theorem:

Let f(x, y) be a continuous function that satisfies a Lipschitz condition. $|f(x,y_1)-f(x,y_2)| \le K|y_1-y_2|$ on a strip defined by a≤x≤b and $-\infty<y<\infty$. If (x_0,y_0) is any point of the strip, then the initial value problem y' = f(x, y), y(x_0) = y_0 has one and only one solution y = y(x) on the interval a ≤x≤ b.

Proof:

We know that every solution of the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0 \qquad \text{......... (1)}$$

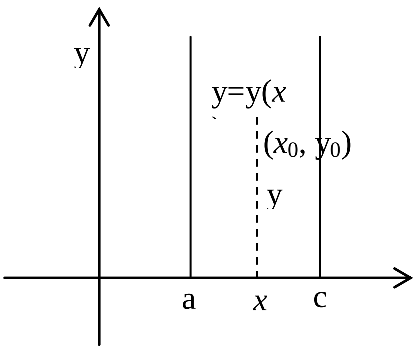
is also a continuous solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)).dt \qquad \text{......... (2)}$$

and conversely.

By successive approximation

We have the sequence of function



$$y_0(x) = y_0$$

$$\left.\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0(t)).dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t)).dt \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t)).dt \end{aligned}\right\} \qquad \text{.... (A)}$$

We observe that $y_n(x)$ is the nth partial sum of the series

$$y_o(x)+(y_1(x)-y_0(x))+(y_2(x)-y_1(x))+\ldots..+(y_n(x)-y_{n-1}(x))+\ldots.. \qquad \text{......... (3)}$$

So the convergence of $\{y_n(x)\}$ is equivalent to the convergence of series (3)

Also,

We know that the series (3) is cgt only if the series.

$|y_0(x)| + |y_1(x) - y_0(x)| + \ldots\ldots + |y_n(x) - y_{n-1}(x)| + \ldots.$ is cgt .... (4)

First we define

M_0, M_1 and M by

$M_0 = |y_0|$

$M_1 = \max |y_1(x)|$

$M = M_0 + M_1$

We find $|y_0(x)| \leq M$ and

$$|y_1(x)| - y_0(x)| \leq |y_1(x)| + |y_0(x)|$$

$$\leq M_1 + M_0$$

$$\therefore |y_1(x) - y_0(x)| \leq M$$

If $x_0 \leq x \leq b$

$$|y_2(x) - y_1(x)| = \left| \int_{x_0}^{x} [f(t, y_1(t) - f(t, y_0(t)] dt \right|$$

$$\leq \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_0(t))| \, dt$$

$$\leq K \int_{x_0}^{x} |y_1(t) - y_0(t)| dt$$

$$\leq K \int_{x_0}^{x} M \, dt$$

$$\leq KM \int_{x_0}^{x} dt$$

$\therefore |y_2(x) - y_1(x)| \leq KM(x - x_0)$

Also,

$$|y_3(x)-y_2(x)| = \left|\int_{x_0}^{x}[f(t,y_2(t)-f(t,y_1(t)]dt\right|$$

$$\leq \int_{x_0}^{x}|f(t,y_2(t))-f(t,y_1(t))|\ dt$$

$$\leq K\int_{x_0}^{x}|y_2(t)-y_1(t)|dt$$

$$\leq K\int_{x_0}^{x}KM\,(t-x_0)\,dt$$

$$\leq K^2M\frac{(x-x_0)^2}{2!}$$

In general

$$|y_n(x)-y_{n-1}(x)| \leq K^{n-1}M\frac{(x-x_0)^{n-1}}{(n-1)!} \qquad \text{.........} \qquad (5)$$

The same argument is also valid for a $\leq x \leq x_0$ provided that $(x\text{-}x_0)$ is replaced by $|x\text{-}x_0|$ is replaced by $|x\text{-}x_0|$.

$$\therefore |y_n(x)-y_{n-1}(x)| \leq K^{n-1}M\frac{(x-x_0)^{n-1}}{(n-1)!} \qquad \text{.........} \qquad (6)$$

Combining (5) and (6) we find that the result holds in the interval a $\leq x\leq$ b, we get

$$|y_n(x)-y_{n-1}(x)| \leq K^{n-1}M\frac{(x-x_0)^{n-1}}{(n-1)!}$$

$$\leq K^{n-1}M\frac{(b-a)^{n-1}}{(n-1)!}\ \forall x \text{ in a} \leq x \leq \text{b}$$

Using the series (4)

$$|y_0(x)|+|y_1(x)-y_0(x)|+|y_2(x)-y_1(x)|+\ldots+|y_n(x)-y_{n-1}(x)|+\ldots+$$

$$\leq M+M+KM(b-a)+K^2M\frac{(b-a)^2}{2!}+$$

$$K^3M\frac{(b-a)^3}{3!}+\ldots+\frac{K^{n-1}M(b-a)^{n-1}}{(n-1)!}+\ldots.$$

The series in the R.H.S.is cgt

∴ The series in the L.H.S is cgt.

∴ The series (3) convergent uniformly on the interval a ≤x ≤b to a limit function y(x)

Let us assume that $\bar{y}(x)$ is also a soln on the same interval.

So $\bar{y}(x)$ is a continuous solution of the integral equation x

$$\therefore \bar{y}(x) = y_0 + \int_{x_0}^{x} f\left(t, \bar{y}(t)\right) dt$$

If A = max $|\bar{y}(x)-y_0|$

Then for $x_0 \le x \le$ b we see that

$$\left|\bar{y}(x) - y_1(x)\right| = \left|\int_{x_0}^{x} \left[f(t, \bar{y}(t) - f(t, y_0(t))\right]\right| dt$$

$$\le K \int_{x_0}^{x} \left|\bar{y}(t) - y_0(t)\right| dt$$

$$\le K \int_{x_0}^{x} A\, dt$$

$$\le K A (x - x_0)$$

Also,

$$\left|\bar{y}(x) - y_2(x)\right| = \left|\int_{x_0}^{x} \left[f(t, \bar{y}(t) - f(t, y_1(t))\right] dt\right|$$

$$\le K \int_{x_0}^{x} \left|\bar{y}(t) - y_1(t)\right| dt$$

$$\le K \int_{x_0}^{x} KA(t - x_0).dt$$

$$\le K^2 A \frac{(x - x_0)^2}{2!} \ldots \text{etc.}$$

In general

$$\left|\bar{y}(x) - y_n(x)\right| \le K^n A \frac{(x - x_0)^n}{n!}$$

The similar result holds for a $\leq x \leq x_0$ for any x in the interval.

∴ We have,

$$\left|\bar{y}(x) - y_n(x)\right| \leq \quad K^n A \frac{|x - x_0|^n}{n!}$$

∴ The similar result holds for a $\leq x \leq$ b.

$$\therefore \left|\bar{y}(x) - y_n(x)\right| \leq \quad K^n A \frac{|x - x_0|^n}{n!}$$

$$= \quad K^2 A \frac{(b-a)^n}{n!}$$

R.H.S.⟶0 as n⟶∞

$\therefore |\bar{y}(x) - y_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$

$\therefore \bar{y}(x) - y_n(x) \longrightarrow 0$

$\therefore \bar{y}(x) = y_n(x)$

But we have, $y_n(x) = y(x)$

$\therefore \bar{y}(x) = y(x)$ for every x in the interval.

Hence the proof.

Problem

1. Let (x_0, y_0) be an arbitrary paint in the plane and consider the initial value problem $y' = y^2$, $y(x_0) = y_0$

Explain why Picard's thm guarantees that this problem has a unique solution on some interval $|x - x_0| \leq h$ $f(x, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous on the entire plane, it is tempting to conclude that this soln is valied for all x. By considering the solution through the points (0,0) and (0,1). S.T. this conclusion is sometimes true and sometimes false, and that therefore the inference is not legitimate.

Solution

Given initial value problem,

$y' = y^2$, $y(x_0) = y_0$

$y = f(x, y), y(x_0) = y_0$

Here $f(x, y) = y^2$, $\frac{\partial f}{\partial y} = 2y$.

Clearly $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all y in the plane for $|x-x_0| \leq h$

∴ By Picard's theorem, the problem has a unique solutionn in $|x-x_0| \leq h$.

Since $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all values of y and $f(x, y)$ and $\frac{\partial f}{\partial y}$ are free from x are tempted to conclude that, the initial value problem has a unique solotionn for all values of x and y (i. e) in the entire plane.

Now, consider the equation.

$y' = y^2$

and examine its soln at (0,0) and (0,1) we have,

$y' = y^2$

$$\frac{dy}{dx} = y^2$$

$$\frac{dy}{y^2} = dx$$

∫ ing

$$-\frac{1}{y} = x + A \qquad \text{......... (1)}$$

At (0, 0) we cannot find the constants A

∴ At the point (0, 0) the soln does not exists At (0, 1)

At (0, 1)

$$(1) \Rightarrow \frac{-1}{1} = 0 + A$$

$$\Rightarrow A = -1$$

$$\therefore -\frac{1}{y} \quad = \quad x-1$$

$$-1 \quad = \quad xy-y$$

$$\Rightarrow xy-y+1 = 0$$

Which is a unique soln.
∴ The solution exists at this point.

Hence our conclusion that, the problem has a unique soln is sometimes true and sometimes false.

So our inference that initial value problem has a unique soln in the entire plane is not legitimate.

2. Show that f(x, y) = $y^{1/2}$
   a) does not satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $0 \leq y \leq 1$.
   b) does satisfy a Lipschitz condition on the rectangle $|x| \leq 1$ and $c \leq y \leq d$ where $o < c < d$.

Solution:

Let f(x, y) = $y^{1/2}$

$$\frac{f(x,y)-f(x,0)}{y-0} = \frac{y^{\frac{1}{2}}-0}{y}$$

$$= \frac{1}{y^{\frac{1}{2}}}$$

Which is not bounded near y = 0

∴ There does not exists K>0

s.t. $|f(x, y)-f(x, 0)| \leq K\,|y-0)|$.

∴ Lipschitz condition is not satisfied.

b) f(x, y) = $y^{1/2}$, $|x| \leq 1$, $y \leq d$ and $0 < c < d$

$$\frac{f(x,y)-f(x,c)}{y-c} = \frac{y^{\frac{1}{2}}-c^{\frac{1}{2}}}{y-c}$$

$$= \frac{1}{y^{\frac{1}{2}} + c^{\frac{1}{2}}}$$

$$\leq \frac{1}{c^{\frac{1}{2}} + c^{\frac{1}{2}}}$$

$$= \frac{1}{2c^{\frac{1}{2}}} = \text{K say}$$

$$\therefore \frac{f(x,y) - f(x,c)}{|y_1 - y_2|} \leq K \quad \forall \text{ y in } c \leq y \leq d.$$

$$\therefore |f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|$$

∴ Lipschitz condition is satisfied.

3. Show that. f(x, y) = x^2|y| satisfies a Lipschitz condition on the rectangle |x|≤ 1 and |y|≤ 1 but that $\frac{\partial f}{\partial y}$ fails to exist at many paints of this rectangle.

Solution:

Let f(x, y) = x^2|y|

$$\frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2} = \frac{x^2 y_1 - x^2 y^2}{y_1 - y_2}$$

$$= \frac{x^2 (y_1 - y_2)}{y_1 - y_2}$$

$$= x^2$$

$$\therefore \left| \frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2} \right| = |x^2| \leq 1$$

$$\therefore |f(x, y_1) - f(x, y_2)| \leq K\,|y_1 - y_2|.$$

∴ Lipschitz condition is satisfied.

Again,

$$\frac{\partial f}{\partial y} = \underset{y \to 0}{Lt} \frac{f(x,y) - f(x,0)}{y - 0}$$

$$= \quad \underset{y \to 0}{Lt} \; \frac{x^2|y| - 0}{y}$$

$$= \quad \underset{y \to 0}{Lt} \; \frac{x^2|y|}{y}$$

$$= \quad x^2\,(\pm 1) \qquad \frac{|y|}{y} = 1 \text{ if } y>0$$

$$= \quad \pm x^2 \qquad \frac{|y|}{y} = -1 \text{ if } y<0$$

Which is not unique.

For, $y > 0, \frac{\partial f}{\partial y} = x^2$

$y < 0, \frac{\partial f}{\partial y} = -x^2$

$y = 0, \frac{\partial f}{\partial y}$ does not exists.

4. Show that $f(x, y) = xy^2$
   a) satisfies a Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$.
   b) does not satisfy a Lipschitz condition on any strip $a \le x \le b$ and $-\infty < y < \infty$

Solution:

Let $f(x, y) = xy2$

$$\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} = \frac{xy_1^2 - xy_2^2}{y_1 - y_2}$$

$$= \frac{x(y_1 + y_2)(y_1 - y_2)}{y_1 - y_2}$$

$$\le \quad b(d+d)$$

$$\le \quad 2bd \quad = \quad K$$

$$\therefore |f(x, y_1) - f(x, y_2)| \quad \le \quad K\,|y_1 - y_2|.$$

∴ Lipschitz condition is satisfied.

b) $$\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2} = \frac{xy_1^2-xy_2^2}{y_1-y_2}$$

$= x(y_1+y_2)$

$=$ not bounded for large values of y.

∴ There does not exists K s.t.

$\therefore |f(x, y_1) - f(x, y_2)| \leq K |y_1-y_2|$.

∴ Lipschitz condition is not satisfied.

5. S.T. $f(x, y) = xy$
   a) satisfies a Lipschtiz condition on any rectangle $c \leq y \leq d$
   b) satisfies a Lipschtiz condition on any strip $a \leq x \leq b$ and $-\infty < y < \infty$.
   c) does not satisfy a Lipschitz condition on the entire plane.

Solution:

a) Let $f(x, y) = xy$

$$\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2} = \frac{xy_1-xy_2}{y_1-y_2}$$

$$= \frac{x(y_1-y_2)}{y_1-y_2}$$

$\leq b = K$ (say)

$\therefore |f(x, y_1) - f(x, y_2)| \leq K |y_1-y_2|$

∴ Lipschitz condition is satisfied.

b) $$\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2} = \frac{xy_1-xy_2}{y_1-y_2}$$

$$= \frac{x(y_1-y_2)}{y_1-y_2}$$

$\leq$ b say (K)

∴ Lipschitz condition is satisfied.

c) $\dfrac{f(x,y_1)-f(x,y_2)}{y_1-y_2} = x$

Which is unbounded for all values of x.

∴ There does not exists K s.t.

∴ $|f(x, y_1) - f(x, y_2)| \leq K |y_1-y_2| \quad -\infty<x<\infty, -\infty< y <\infty$

∴ Lipschitz condition is not satisfied in the entire plane.

6. Consider the initial value problem.

   $y' = |y|, y(x_0) = y_0$.

   a) For what points (x_0, y_0) does Picard theorem imply that this problem has unique solution on some interval $|x-x_0|\leq h$.

   b) For what points (x_0, y_0) does this prob actually have a unique solution in some interval. $|x-x_0|\leq h$.

Solution:

Let $y' = |y|, y(x_0) = y_0$.

Clearly $f(x, y) = |y|$ is continuous in the plane.

$$\frac{\partial f}{\partial y} = \underset{y\to 0}{Lt} \frac{f(x,y)-f(x,0)}{y-0}$$

$$= \underset{y\to 0}{Lt} \frac{|y|-0}{y}$$

Which is not unique.

∴ $\dfrac{\partial f}{\partial y}$ does not exists at y = 0.

Hence by Picard's thm, a unique soln exists for all points (x_0, y_0) except those with $y_0= 0$.

Let us examine the solution at points where $y_0 = 0$, using Lipschitz condition.

We have,

$$\frac{f(x,y)-f(x,0)}{y-0} = \frac{|y|-0}{y}$$

$$= \pm 1$$

$$\frac{f(x,y)-f(x,0)}{|y-0|} = 1 = K \text{ (say)}$$

∴ We get,

∴$|f(x, y) - f(x, 0)| \leq K |y-0|$

∴ Lipschitz condition is satisfied.

Even at points where $y_0 = 0$, there is a unique solution for the problem.

Hence this has unique soln actually at all points (x_0, y_0).

Linear Systems:

Equations of the form

$$\left.\begin{aligned} \frac{dx}{dy} &= F(t,x,y) \\ \frac{dy}{dt} &= G(t,x,y) \end{aligned}\right\} \quad \text{......... (1)}$$

are said to be a system of simultaneous equation of first order:

System of linear equations

The equation of the form,

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t) \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t) \end{aligned}\right\} \quad \text{......... (2)}$$

Where a_1, b_1, f_1, a_2, b_2, f_2 are continuous functions in any closed interval [a, b]

(i.e) $a \leq x \leq b$

The equ (2) are said to be a system of linear equations

If $f_1(t)$ and $f_2(t)$ are identically zero, then the equation reduces to

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \quad \text{......... (3)}$$

System of equ (3) is called homogeneous equation and system of equation (2) is called non-homogeneous equation.

Verify that the linear system of equation

$$\left.\begin{aligned} \frac{dy}{dt} &= 4x - y \\ \frac{dy}{dt} &= 2x - y \end{aligned}\right\}$$

has both $\begin{cases} x = e^{3t} \\ y = e^{3t} \end{cases}$ and $\begin{cases} x = e^{2t} \\ y = 2e^{2t} \end{cases}$

are solutions in any closed interval.

Solution:

Let $x = e^{3t}$, $y = e^{3t}$

$\frac{dy}{dt} = 3e^{3t}$, $\qquad \frac{dy}{dt} = 3e^{3t}$

$4x\text{-y} = 4e^{3t}-e^{3t}$ $\qquad 2x\text{+y} = 2e^{3t}+e^{3t}$

$= 3e^{3t}$ $\qquad = 3e^{3t}$

$\frac{dx}{dy} = 4x - y$ $\qquad \frac{dy}{dt} = 2x + y$

Thus, $\begin{cases} x = e^{3t} \\ y = e^{3t} \end{cases}$ is a soln of the gn equation.

Let $x = e^{3t}$, $\qquad y = e^{3t}$

$\frac{dx}{dt} = 2e^{2t}$ $\qquad \frac{dy}{dt} = 4e^{2t}$

$4x\text{-y} = 4e^{2t}-2e^{2t}$ $\qquad 2x\text{+y} = 2e^{2t}+2e^{2t}$
$= 2e^{2t}$ $\qquad = 4e^{2t}$

$\frac{dx}{dy} = 4x - y$ $\qquad \frac{dy}{dy} = 4e^{2t}$

$\therefore \begin{cases} x = e^{2t} \\ y = 2e^{2t} \end{cases}$ is a solution of the given equation

Theorem:

If the homogeneous system. $\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}$ has two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$

and $\begin{cases} x = x^2(t) \\ y = y_2(t) \end{cases}$ on [a, b], then $\begin{cases} x = c_1x_1(t)+c_2x_2(t) \\ y = c_1y_1(t)+c_2y_2(t) \end{cases}$ is also a soln. on [a, b] forany constants c_1 and c_2.

Proof:

Given $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ is a solution of

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \qquad \text{......... (1)}$$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_1(t)x + b_1(t)y_1 \\ \frac{dy_1}{dt} &= a_2(t)x + b_2(t)y_1 \end{aligned}\right\} \qquad \text{......... (2)}$$

///rly since $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ is a soln, we get

$$\left.\begin{aligned} \frac{dx_2}{dt} &= a_1(t)x_2 + b_1(t)y_2 \\ \frac{dy_2}{dt} &= a_2(t)x_2 + b_2(t)y_2 \end{aligned}\right\} \qquad \text{......... (3)}$$

Take $x = c_1x_1(t)+c_2x_2(t)$

$$\frac{dx}{dt} = c_1\frac{dx_1}{dt} + c_2\frac{dx_2}{dt}$$

$= c_1[a_1(t)x_1+b_1(t)y_1] + c_2 [a_1(t)x_2+b_1(t)y_2]$

$= a_1(t) [c_1x_1+c_2x_2] + b_1(t) [c_1y_1+c_2y_2]$

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y \qquad \text{......... (I)}$$

///rly Take $y = c_1y_1(t) + c_2y_2(t)$

We get $\frac{dy}{dt} = c_1\frac{dy_1}{dt} + c_2\frac{dy_2}{dt}$

$= c_1[a_2(t)x_1+b_2(t)y_1] + c_2 [a_2(t)x_2+b_2(t)y_2]$

$= a_2(t) [c_1x_1+c_2x_2] + b_2(t) [c_1y_1+c_2y_2]$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y \qquad \text{......... (II)}$$

Equations I and II together gives the system of equations, which are satisfied by

$x = c_1x_1(t)+c_2x_2(t)$

$y = c_1y_1(t)+c_2y_2(t)$ be the solution of the equation.

Hence the proof.

Theorem:

If the system of equation $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$ has, $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ as a solution of the interval [a, b], then $\begin{cases} x = c_1x_1(t)+c_2x_2(t) \\ y = c_1y_1(t)+c_2y_2(t) \end{cases}$ is the general solution, if the Wronskian of the solution, does not Vanish on the interval [a, b].

Proof:

If $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ are the solutions of the given system of equation then by the previous theorem,

$$x = c_1x_1(t)+c_2x_2(t)$$
$$y = c_1y_1(t)+c_2y_2(t) \quad \text{......... (1)}$$

Where c_1 and c_2 are constants is also a solution of the system on the interval [a, b].

If (1) is the general solution, them the constants c_1 and c_2 are unique.

Let us assume the initial condition,

When $t = t_0$, $x = x_0$, $y = y_0$

$$\left.\begin{aligned} c_1x_1(t_0)+c_2x_2(t_0) &= x_0 \\ c_1y_1(t_0)+c_2y_2(t_0) &= y_0 \end{aligned}\right\} \quad \text{......... (2)}$$

The equation (2) have unique solution, if the coefficient determinant does not vanish.

$$\therefore \quad \begin{vmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{vmatrix} \neq 0$$

Since t_0 is arbitrary, we find

$$\begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix} \neq 0$$

Wronskian, $W \neq o$

Hence the proof.

Problem:

The system of equation

$$\frac{dx}{dt} = 4x - y$$

$$\frac{dy}{dt} = 2x + y$$

We have two solutions. $\begin{cases} x_1 = e^{3t} \\ y_1 = e^{3t} \end{cases}$ and $\begin{cases} x_2 = e^{2t} \\ y_2 = 2e^{2t} \end{cases}$, we get.

Wronskian, W = $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

$= \begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix}$

$= 2e^{5t} - e^{5t}$

$= e^{5t} \neq 0.$

∴ For the given equation.

$x = c_1e^{3t}+c_2e^{2t}$

$y = c_1e^{3t}+2c2e^{2t}$ is a general solution.

Note:

In the general solution if the constants c_1 and c_2 are evaluated using initial condition we get a particular solution.

Note:

Wronskian of the solution $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$

$x = x_2(t)$
$y = y_2(t)$

(i.e) W $= \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$

$= \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

∴ W $= x_1y_2 - y_1x_2$

Theorem:

If W(t) is the Wronskian of the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the system of equation. $\frac{dx}{dt} = a_1(t)x + b_1(t)y$ on [a, b], then W(t) is either identically zero or

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y$$

nowhere zero on [a,b].

Proof:

Given that $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ is the soln of the equ $\begin{cases} \dfrac{dx}{dt} = a_1(t)x + b_1(t)y \\ \dfrac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$

∴ We get

$$\frac{dx_1}{dt} = a_1(t)x_1 + b_1(t)y_1$$

$$\frac{dy_1}{dt} = a_2(t)x_1 + b_2(t)y_1$$

|||rly since $\begin{cases} x = x_2(t) \\ y = y_1(t) \end{cases}$ is a soln, we get.

$$\frac{dx_2}{dt} = a_1(t)x_2 + b_1(t)y_2$$

$$\frac{dy_2}{dt} = a_2(t)x_2 + b_2(t)y_2$$

Now,

$$\frac{dx_1}{dt}y_2 - \frac{dx_2}{dt}y_1 = [a_1(t)x_1 + b_1(t)y_1]y_2 - [a_1(t)x_2 + b_1(t)y_2]y_1$$

$$= a_1(t)x_1y_2 + b_1(t)y_1y_2 - a_1(t)x_2y_1 - b_1(t)y_1y_2$$

$$= a_1(t)\,[x_1y_2 - x_2y_1] \qquad \text{......... (1)}$$

|||rly

$$\frac{dy_2}{dt}x_1 - \frac{dy_1}{dt}x_2 = x_1[a_2(t)x_2 + b_2(t)y_2] - x_2[a_2(t)x_1 + b_2(t)y_1]$$

$$= a_2(t)x_1x_2 + b_2(t)x_1y_2 - a_2(t)x_1x_2 - b_2(t)y_1x_2$$

$$= b_2(t)\,[x_1y_2 - y_1x_2] \qquad \text{......... (2)}$$

(1) + (2)

$$\left[\frac{dx_1}{dt}y_2 - \frac{dx_2}{dt}y_1\right] + \left[\frac{dy_2}{dt}x_1 - \frac{dy_1}{dt}x_2\right] = \quad [a_1(t)+b_2(t)](x_1y_2 - x_2y_1)$$

$$\left[\frac{dx_1}{dt}y_2 + \frac{dy_2}{dt}x_1\right] - \left[\frac{dx_2}{dt}y_1 + \frac{dy_1}{dt}x_2\right] = \quad [a_1(t)+b_2(t)](x_1y_2 - x_2y_1)$$

$$\Rightarrow \frac{d}{dt}(x_1\,y_2) - \frac{d}{dt}(x_2\,y_1) \quad = \quad (a_1(t)+b_2(t))(x_1y_2 - x_2y_1)$$

$$\Rightarrow \frac{d}{dt}(x_1\,y_2 - x_2\,y_1) \quad = \quad (a_1(t)+b_2(t))(x_1y_2 - x_2y_1) \quad \text{......... (3)}$$

Wronskian is W $= \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

$= x_1y_2 - y_1x_2$

$\therefore (3) \Rightarrow$

$$\frac{dw}{dt} = a_1(t) + b_2(t)W$$

$$\therefore \frac{dw}{W} = [a_1(t) + b_2(t)]dt$$

$\int$ ing

log W $=$ $\int[a_1(t)+b_2(t)]dt$ + log C

log W $=$ $\log ce^{\int[a_1(t)+b_2(t)]dt}$

$\therefore$ W $=$ $ce^{\int[a_1(t)+b_2(t)]dt}$ for some constant c.

We observe that, the exponential factor in the above is never zero.

Therefore the Wronskian W can be zero only if it is identically zero. Otherwise it is never zero on the interval [a, b].

Hence the proof.

Dependent and Independent solutions.

Consider the system

$$\frac{dx}{dy} = a_1x + b_1y$$

$$\frac{dy}{dt} = a_2x + b_2y$$

Let $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ be two solutions of the system of equation

The two solutions are said to be linearly dependent if one is a constant multiple of the other.

(i.e) if $x_2(t) = Kx_1(t)$

$y_2(t) = Ky_1(t)$. Where K is a constant.

∴ Two dependent solns will be of the form $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = Kx_1(t) \\ y = Ky_1(t) \end{cases}$

If one solution is not a constant multiple of the other, then the solutions are said to be linearly independent.

Further consider the equation

$c_1x_1 + c_2x_2 = 0$

$c_1y_1 + c_2y_2 = 0$

The solns are independent iff $c_1 = 0$, and $c_2 = 0$

If one or both of c_1 and c_2 are non zero, then the solutions are linearly dependent.

Theorem:

For the homogeneous system of equ $\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y \end{cases}$, the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ are L.I on [a, b] Then $\begin{cases} x = c_1x_1 + c_2x_2 \\ y = c_1y_1 + c_2y_2 \end{cases}$ will be the general solution of the system.

Proof:

If $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ are tow solutions of the system of equ.

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \qquad \text{......... (1)}$$

then $\left.\begin{aligned} x &= c_1x_1 + c_2x_2 \\ y &= c_1y_1 + c_2y_2 \end{aligned}\right\}$ .... (2)

Will be a general solution if the wronskian of the solution $W \neq 0$.

Suppose the given solution are L.D. then

$x_2(t) = Kx_1(t)$

$y_2(t) = Ky_1(t)$ where k is a constant.

$$W = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & Kx_1 \\ y_1 & Ky_1 \end{vmatrix}$$

$$= Kx_1y_1 - Kx_1y_1$$

$$W = 0$$

∴ Equation (2) is not a general solution of the system of equation (1).

Suppose the Wronskian

(i.e) $W = 0$

$c_1x_1 + c_2x_2 = 0$

$c_1y_1 + c_2y_2 = 0$

$\therefore c_2x_2 = -c_1x_1$

$$x_2 = -\frac{c_1}{c_2}x_1$$

$$x_2 = K\, x_1$$

$$y_2 = -\frac{c_1}{c_2}y_1$$

$$y_1 = K\, y_1$$

∴ The solutions are linearly dependent.

Thus we find (2) is not a general solution if the solutions are dependent.

∴ Equation (2) will be a general solution the solutions are L.I.

4. Problem:

Let the second order linear equation

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + Q(t)x = 0 \qquad \text{......... (1)}$$

be reduced to the system

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -Q(t)x - p(t)y \qquad \text{......... (2)}$$

If $x_1(t)$ and $x_2(t)$ are the solutions of equation (1) and if $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ are the corresponding solution of (2). S.T. the Wronskian of (1) is same as the Wronskian of the solution (2).

Proof:

Consider the second order equation

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + Q(t)x = 0 \qquad \text{......... (1)}$$

If x_1 and x_2 are two solutions of the equation (1). Then the Wronskian

$$W = \begin{vmatrix} x_1 & x_2 \\ x_1^1 & x_2^1 \end{vmatrix}$$

$$= x_1 x_2^1 - x^2 x_1^1 \qquad \text{......... (3)}$$

Put $\frac{dx}{dt} = y$

$$\therefore \frac{d^2x}{dt^2} = \frac{dy}{dt}$$

∴ Equ (1) reduces to

$$\frac{dy}{dt} + p(t)y + Q(t)x = 0$$

$$\therefore \frac{dx}{dt} = -p(t)y - Q(t)x$$

∴ We get, the system of equation

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dx}{dt} &= -p(t)y - Q(t)x \end{aligned}\right\} \qquad \text{......... (2)}$$

If $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ are two solutions.

Then Wronskian, $W_1 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

$$= x_1 y_2 - x_2 y_1$$

We have, $\frac{dx}{dt} = y$

$$\frac{dx_1}{dt} = y_1 \qquad \frac{dx_2}{dt} = y_2$$

$$\therefore y_1 = x_1^1 \qquad y_2 = x_2 y^1$$

Sub in W_1

$\therefore W_1 = x_1x_2^1 - x_2x_1^1$

$= W$

$\therefore$ The two wronskian are the same.

5. (a) S.T. $\begin{cases} x = e^{4t} \\ y = e^{4t} \end{cases}$ and $\begin{cases} x = e^{-2t} \\ y = -e^{-2t} \end{cases}$ are solutions of the homogeneous system,

$\frac{dx}{dt} = x + 3y$, $\frac{dy}{dt} = 3x + y$.

(b) Show in two ways that the given solution of the system in (a) are L.I on every closed interval and write the general solution of this system.

(c) Find the particular solution $x = x(t)$, $y = y(t)$ of this system for which $x(0) = 5$ and $y(0) = 1$.

Solution:

Consider $\begin{cases} x = e^{4t} \\ y = e^{4t} \end{cases}$

$\therefore \frac{dx}{dt} = 4e^{4t}$

$\frac{dy}{dt} = 4e^{4t}$

$x + 3y = e^{4t} + 3.e^{4t}$

$= 4e^{4t}$

$\therefore \frac{dx}{dt} = x + 3y$

$3x + y = 3e^{-4t} + e^{4t}$

$= 4e^{4t}$

$\therefore \frac{dy}{dt} = 3x + y$

$\therefore \begin{cases} x = e^{4t} \\ y = e^{4t} \end{cases}$ is a solution of the given system of equation

Now consider,

$x = e^{-2t}$, $y = -e^{-2t}$

$$\frac{dx}{dt} = -2e^{-2t} \qquad \frac{dy}{dt} = 2e^{-2t}$$

$x+3y = e^{-2t} + 3(-e^{-2t})$

$= -2e^{-2t}$

$$\therefore \quad \frac{dx}{dt} = -2e^{-2t}$$

$3x+y = 3e^{-2t} - e^{-2t}$

$= 2e^{-2t}$

$$\therefore \frac{dy}{dt} = 3x + y$$

$\therefore \begin{cases} x = e^{-2t} \\ y = -e^{-2t} \end{cases}$ is the solution of the given system of equation.

b) Wronskian W $= \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$

$= \begin{vmatrix} e^{4t} & e^{-2t} \\ e^{4t} & -e^{-2t} \end{vmatrix}$

$= -e^{4t}e^{-2t} - e^{4t}e^{-2t}$

$= -e^{2t} - e^{2t}$

$= -2e^{2t} \neq 0$

$\therefore$ The solutions are linearly independent.
Again consider the equation.

$c_1x_1+c_2x_2 = 0$

$c_1y_1+c_2y_2 = 0$

(i.e) $c_1e^{4t}+c_2e^{-2t} = 0$

$c_1e^{4t} - c_2e^{-2t} = 0$

This may be written as,

$$c_1e^{bt} + c_2 = 0$$

$$c_1e^{6t} - c_2 = 0$$

$$2c_2 = 0$$

$$\Rightarrow c_2 = 0$$

$$c_1e^{6t} + 0 = 0$$

$$\Rightarrow c_1 = 0$$

∴ The solutions are linearly independent.

Since $\begin{cases} x = e^{4t} \\ y = e^{4t} \end{cases}$ and $\begin{cases} x = e^{-2t} \\ y = -e^{-2t} \end{cases}$ are L.I.

∴ The general solution can be taken as.

$$x = c_1e^{4t} + c_2e^{-2t}$$

$$y = c_1e^{4t} - c_2e^{-2t}$$

c) To find the particular solution corresponding to $x(0) = 5$, $y(0) = 1$

(i.e) When $t = 0$, $x = 5$, $y = 1$

We get, $5= c_1e^0+c_2e^0$

$1 = c_1e^0 - c_2e^0$

$$c_1 + c_2 = 5$$

$$c_1 - c_2 = 1$$

$$2c_1 = 6$$

$$c = 3$$

$$2c_2 = 4$$

$$c_2 = 2$$

∴ The Particular solution as

$$\begin{cases} x = c_1x_1 + c_2x_2 \\ y = c_1y_1 + c_2y_2 \end{cases}$$

$\therefore x = 3e^{4t} + 2e^{-2t}$

$y = 3e^{4t} - 2e^{-2t}$

7. Obtain the solution of the homogenous system $\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$

a) By differentiating the first equation w.r.to. 't' and eliminating y.
b) By differentiating the second equation w.r.to. 't' and eliminating *x*.

Solution:

Given equation is

$$\frac{dx}{dt} = x + 2y \qquad \text{......... (1)}$$

$$\frac{dy}{dt} = 3x + 2y \qquad \text{......... (2)}$$

Diff (1)

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2\frac{dy}{dt}$$

$$= \frac{dx}{dt} + 2(3x + 2y)$$

$$= \frac{dx}{dt} + (6x + 4y)$$

$$= \frac{dx}{dt} + 6x + 2\left[\frac{dx}{dt} - x\right]$$

$$= 3\frac{dx}{dt} + 4x$$

$$\therefore \frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 4x = 0 \qquad \text{......... (3)}$$

Auxillary equation is

$$m^2 - 3m - 4 = 0$$

$$(m-4)(m+1) = 0$$

$$\therefore m = 4, -1.$$

$\therefore$ Solution of (3) is

$$x = c_1e^{4t} + c_2e^{-t}$$

$$\therefore \frac{dx}{dt} = 4c_1e^{4t} - c_2e^{-t}$$

From (1)

$$2y = \frac{dx}{dt} - x$$

$$= (4c_1e^{4t} - c_2e^{-t}) - (c_1e^{4t} + c_2e^{4t})$$

$$= 3c_1e^{4t} - 2c2e^{-t}$$

$$\therefore y = \frac{3}{2}c_1e^{4t} - c_2e^{-t}$$

Theorem:

If the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous system

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y$$

are L.I. on [a, b] and if $\begin{cases} x = x_p(t) \\ y = y_p(t) \end{cases}$ is any particular solution of the nonhomogeneous system.

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$$

on this interval, then $\begin{cases} x = c_1x_1(t) + c_2x_2(t) + x_p(t) \\ y = c_1y_1(t) + c_2y_2(t) + y_p(t) \end{cases}$ is the general solution of the non-homogeneous system on [a, b].

Proof:

Given $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ are independent solutions of

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y\end{aligned}\right\} \quad \text{......... (A)}$$

∴ The general solution of the homogenous system (A) can be taken as.

$$x = c_1x_1(t)+c_2x_2(t)$$
$$y = c_1y_1(t)+c_2y_2(t)$$

Let $\left\{\begin{array}{l} x = x(t) \\ y = y(t)\end{array}\right.$ be the solutions of the non-homogeneous equation.

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t) \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t)\end{aligned}\right\} \quad \text{......... (B)}$$

∴ We get,

$$\frac{dx(t)}{dt} = a_1(t)x(t) + b_1(t)y(t) + f_1(t)$$

$$\frac{dy(t)}{dt} = a_2(t)x(t) + b_2(t)y(t) + f_2(t)$$

Also, $\left\{\begin{array}{l} x = x_p(t) \\ y = y_p(t)\end{array}\right.$ is given to be a particular solution of the non-homogeneous system (B).

$$\therefore \frac{dx_p(t)}{dt} = a_1(t)x_p(t) + b_1(t)y_p(t) + f_1(t)$$

$$\frac{dy_p(t)}{dt} = a_2(t)x_p(t) + b_2(t)y_p(t) + f_2(t)$$

Take $x = x(t) - x_p(t)$

$y = y(t) - y_p(t)$

We get,

$$\frac{dx}{dt} = \frac{d}{dt}\left(x(t) - x_p(t)\right)$$

$$= \frac{dx(t)}{dt} - \frac{dx_p(t)}{dt}$$

$$= a_1(t)\, x(t) + b_1(t)\, y(t) + f_1(t) - [a_1(t)\, x_p(t) + b_1(t)\, y_p(t) + f_1(t)]$$

$$\left.\begin{aligned} \frac{d}{dt}[x(t)-x_p(t)] &= a_1(t)\,[x(t) - x_p(t)] + b_1(t)\,[y(t) - y_p(t)] \\ \frac{d}{dt}[y(t)-y_p(t)] &= a_2(t)\,[x(t) - x_p(t)] + b_2(t)\,[y(t) - y_p(t)] \end{aligned}\right\} \quad \text{......... (C)}$$

From (C) we find

$$\begin{cases} x = x(t) - x_p(t) \\ y = y(t) - y_p(t) \text{ is a soln of (A)} \end{cases}$$

$$\therefore x(t) - x_p(t) = c_1x_1 + c_2x_2$$

$$y(t) - y_p(t) = c_1y_1 + c_2y_2$$

$$\Rightarrow \quad x(t) = c_1x_1 + c_2x_2 + x_p(t)$$

$$y(t) = c_1y_1 + c_2y_2 + y_p(t)$$

is a general solution of the non-homosystem.

6. (a) S.T. $\begin{cases} x = 2e^{4t} \\ y = 3e^{4t} \end{cases}$ and $\begin{cases} x = e^{-t} \\ y = -e^{-t} \end{cases}$ are solutions of homo-system.

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = 3x + 2y$$

(b) Show in two ways that the given solution of the system in (a) are L.I. in every closed interval and write the general solution of the system.

(c) S.T. $\begin{cases} x = 3t-2 \\ y = -2t+3 \end{cases}$ is a particular solution of the non-homogeneous system.

$$\frac{dx}{dt} = x + 2y + t - 1$$

$$\frac{dy}{dt} = 3x + 2y - 5t - 2$$ and write the general solution of the system.

Solution:

The given homo system is

$$\left.\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 3x + 2y \end{aligned}\right\} \qquad \text{......... (A)}$$

Take $x = 2e^{4t}$

$y = 3e^{4t}$

$\frac{dx}{dt} = 8e^{4t}$

$x+2y = 2e^{4t} + 2.3e^{4t}$

$= 8e^{4t}$

$\therefore \frac{dx}{dt} = x + 2y$

Also, $y = 3e^{4t}$

$\frac{dy}{dt} = 12e^{4t}$

$3x+2y = 3.2.e^{4t} + 2.3e^{4t}$

$= 12\ e^{4t}$

$\therefore \frac{dy}{dt} = 3x + 2y$

$\therefore x = 2e^{4t}$

$y = 3e^{4t}$ is a solution of (A)

Again take $\begin{cases} x = e^{-t} \\ y = -e^{-t} \end{cases}$

$\frac{dx}{dt} = -e^{-t}$

$x+2y = e^{-t} - 2e^{-t}$

$= -e^{-t}$

$\frac{dx}{dt} = x+2y$

$y = -e^{-t}$

$\therefore \frac{dy}{dt} = e^{-t}$

$3x+2y = 3e^{-t} + 2(-e^{-t})$

$= e^{-t}$

$\therefore \left\{ \begin{array}{l} x = e^{-t} \\ y = -e^{-t} \end{array} \right.$ is a solution of (A)

(b) Consider the Wronskina.

$$W = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= \begin{vmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{vmatrix}$$

$= -2e^{3t} - 3e^{3t}$

$= -5e^{3t}$

$\therefore W \neq 0.$

Again consider the equation

$c_1x_1 + c_2x_2 = 0$

$c_1y_1 + c_2y_2 = 0$

$c_1 2e^{4t} + c_2e^{-t} = 0$

$c_1 3e^{4t} - c_2e^{-t} = 0$

Which is same as,

$$2c_1e^{5t} + c_2 = 0$$

$$3c_1e^{5t} - c_2 = 0$$

$$5c_1e^{5t} = 0$$
$$\Rightarrow c_1 = 0$$

Also $c_2 = 0$

$\therefore c_1 = c_2 = 0$

$\therefore$ The solns $\begin{cases} x = 2e^{4t} \\ y = 3e^{4t} \end{cases}$ and $\begin{cases} x = e^{-t} \\ y = -e^{-t} \end{cases}$ are L.I.

So the general solution of the system (A) can be taken as.

$$x = c_1x_1 + c_2x_2$$

$$y = c_1y_1 + c_2y_2$$

$$\therefore \quad x = 2c_1e^{4t} + c_2e^{-t}$$

$$y = 3c_1e^{4t} - c_2e^{-t}$$

(c) We have to prove $\begin{cases} x_p = 3t-2 \\ y_p = -2t+3 \end{cases}$ is a particular solution of.

$$\left.\begin{aligned} \frac{dx}{dt} &= x + 2y + t - 1 \\ \frac{dy}{dt} &= 3x + 2y - 5t - 2 \end{aligned}\right\} \quad \text{......... (B)}$$

$x_p(t) = 3t-2 \qquad y_p(t) = -2t+3$

$\dfrac{dx_p(t)}{dt} = 3 \qquad \dfrac{dy_p(t)}{dt} = -2$

$$x_p+2y_p+t-1 = 3t-2+2(-2t+3)+t-1$$

$$= 3t-2-4t+6+t-1$$

$$= 4t-4t+6-3$$

$$= 3$$

$$\therefore \frac{dx_p(t)}{dt} = x_p+2y_p+t-1 \qquad \text{......... (1)}$$

$3x_p+2y_p-5t-2$

$$= 3(3t-2) + 2(-2t+3)-5t-2$$

$$= 9t-6-4t+6-5t-2$$

$$= 9t-9t-2$$

$$= -2$$

$$\therefore \frac{dy_p(t)}{dt} = 3x_p+2y_p-5t-2 \qquad \text{......... (2)}$$

∴ From (1) and (2) we find

$$\begin{cases} x_p = 3t-2 \\ y_p = -2t+3 \end{cases}$$ is a particular solution of the non-homogeneous system (B).

Hence the general solution of (B) is

$$x = c_1x_1+c_2x_2+x_p$$

$$y = c_1y_1+c_2y_2+y_p$$

(i.e) $x = 2c_1e^{4t}+c_2e^{4t}+c_2e^{-t}+3t-2$

$y = 3c_1e^{4t} - c_2e^{-t}-2t+3$

Problem

Find the general solution of the system

(a) $\frac{dx}{dt} = x, \ \frac{dy}{dt} = y$

(b) S.T. any second order equation obtained from the system in (a) is not, equivalent to this system in the sense that it has solution that are not part of any solution of the system Thus although higher order equations are equivalent to systems, the reverse is not true, and system are more general.

Solution:

Given, $\frac{dx}{dt} = x$

$\frac{dy}{dt} = y$

$\frac{dx}{x} = dt$

$\int \frac{dx}{x} = \int dt$

$\log x = t + \log c_1$

$\log x = \log c_1 e^t$

$x = c_1 e^t$

$\frac{dy}{y} = dt$

$\int \frac{dy}{y} = \int dt$

$\log y = t + \log c_2$

$y = c_2 e^t$

$\therefore \quad x = c_1 e^t$

$y = c_2 e^t$ is the soln of the system.

Again, consider,

$\frac{dx}{x} = x$

$\frac{d^2 x}{dt^2} = \frac{dx}{dt}$

$(D^2 - D)x = 0$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, \; m = 1$

Solution $x = c_1e^t + c_2e^0$

$x = c_1e^t + c_2$

We find $x = 1$ not a solution of $\frac{dx}{dt} = x$

∴ The solution of the second order equation contains a solution which is not a solution of the system.

But the solutions of the system are in the solution of second order equation.

So we conclude, although the second order equation is equivalent to the system. The system is not equivalent to the second order equation in the above sense.

Solutions of Homogeneous equation with constant coefficients

To solve the system of the equation

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1x + b_1y \\ \frac{dy}{dt} &= a_2x + b_2y \end{aligned}\right\} \qquad \text{.........} (1)$$

Let us assume,

$x = Ae^{mt}$

$y = Be^{mt}$ as a soln.

Substituting in the equation

We get,

$Am\,e^{mt} = a_1Ae_{mt} + b_1Be^{mt}$

$Bm\,e^{mt} = a_2\,Ae^{mt} + b_2\,Be^{mt}$

Cancelling e^{mt}

$Am = a_1A + b_1B$

$Bm = a_2A + b_2B$

Thus we get, two equations in A and B.

$$\left.\begin{aligned}(a_1-m) A+b_1B &= 0\\ a_2A+(b_2-m)B &= 0\end{aligned}\right\} \qquad \text{.........} (2)$$

Clearly A = 0, B = o, are solution of the equation (2)

These are trivial solution

We know the equation (2) will have non-trivial solution if

$$\begin{vmatrix} a_1-m & b_1 \\ a_2 & b_2-m \end{vmatrix} = 0$$

$$(a_1m)(b_2-m) - a_2b_1 = 0$$

$$a_1b_2 - mb_2 - ma_1 + m^2 - a_2b_1 = 0$$

$$m^2-m(a_1+b^2)+(a_1b_2-a_2b_1) = 0 \qquad \text{.........} (3)$$

This equation (3) is called auxiliary equation of the system.

Solving this equation we get two values of m (say m_1 and m_2)

Sub $m = m_1$ in equation (2) we get a set of values for A and B.

Let them be A_1 and B_1.

The corresponding solution of the system is

$$x_1 = A_1e^{m_1t}$$
$$y_1 = B_1e^{m_1t}$$

|||rly sub $m = m_2$ in the equation (2)

We get a set of values A_2, B_2 of A and B.

The corresponding solution is

$$x_2 = A_2e^{m_2t}$$

$$y_2 = B_2e^{m_2t}$$

Hence the system is solved.

The roots of the auxilliary equation.

$m^2-(a_1+b_2)m+(a_1b_2-a_2b_1) = 0$ may be

(i) real and distinct
(ii) real and equal
(iii) complex

Case (i)

Roots of auxillary equation are real and district.

Let them be m_1 and m_2.

The solutionn of the system are.

$$\begin{cases} x_1 = A_1 e^{m_1 t} \\ y_1 = B_1 e^{m_1 t} \end{cases} \text{ and } \begin{cases} x_2 = A_2 e^{m_2 t} \\ y_2 = B_2 e^{m_2 t} \end{cases}$$

Case (ii)

Roots of auxillary equations are real and equal.

One solution of the system is

$$\begin{cases} x_1 = A_1 e^{m_1 t} \\ y_1 = B_1 e^{m_1 t} \end{cases}$$

Now we have to find another independent soln.

Let us assume that, the solution be

$$x_2 = (A_1 + A_2 t)e^{m_1 t}$$

$$y_2 = (B_1 + B_2 t)e^{m_1 t}$$

We have to find A_1, B_1 and A_2, B_2.

We know, $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ is a solution of the system

$$\left.\begin{aligned}\frac{dx_2}{dt} &= a_1x_2 + b_1y_2 \\ \frac{dy_2}{dt} &= a_2x_2 + b_2y_2\end{aligned}\right\} \quad \text{......... (4)}$$

Now,

$$x_2 = (A_1 + A_2t)e^{m_1t}$$

$$\frac{dx_2}{dt} = (A_1 + A_2t)m_1e^{m_1t} + A_2e^{m_1t}$$

$$y_2 = (B_1 + B_2t)e^{m_1t}$$

$$\frac{dy_2}{dt} = (B_1 + B_2t)m_1e^{m_1t} + B_2e^{m_1t}$$

Sub in (4)

$$(A_1 + A_2t)m_1e^{m_1t} + A_2e^{m_1t} = a_1(A_1 + A_2t)e^{m_1t} + b_1(B_1 + B_2t)e^{m_1t}$$

$$(B_1 + B_2t)m_1e^{m_1t} + B_2e^{m_1t} = a_2(A_1 + A_2t)e^{m_1t} + b_2(B_1 + B_2t)e^{m_1t}$$

Cancelling e^{m_1t} in both sides

$$(A_1 + A_2t)m_1 + A_2 = a_1(A_1 + A_2t) + b_1(B_1 + B_2t)$$

$$(B_1 + B_2t)m_1 + B_2 = a_2(A_1 + A_2t) + b_2(B_1 + B_2t)$$

Equating the constant term and the coefficient of t,

$$\left.\begin{aligned}A_1m_1+A_2 &= a_1A_1 + b_2B_1 \\ A_2m_1 &= a_1A_2+b_1B_2 \\ B_1m_1+B_2 &= a_2A_1+b_2B_1 \\ B_2m_1 &= a_2A_2+b_2B_2\end{aligned}\right\} \quad \text{......... (5)}$$

Solving the equation (5) we get

A_1, A_2, B_1, B_2, and hence the second solution is.

$$x_2 = (A_1 + A_2t)e^{m_1t}$$

$$y_2 = (B_1 + B_2 t)e^{m_1 t}$$

Hence the general solution is

$$x = c_1x_1+c_2x_2$$

$$y = c_1y_1+c_2y_2$$

Case (iii)

Roots of auxillary equations are complex.

If m_1 and m_2 are distinct complex numbers, then they can be written in the form a±ib, where 'a' and 'b' are real numbers, and $b \neq 0$.

$$\left\{\begin{array}{l} x = A_1^* e^{(a+ib)t} \\ y = B_1^* e^{(a+ib)t} \end{array}\right. \quad \text{and} \quad \begin{array}{l} x = A_2^* e^{(a-ib)t} \\ y = B_2^* e^{(a-ib)} \end{array}$$ These are complex valued solns.

If we express the numbers A_1^* and B_1^* in the standard form

$A_1^* = A_1+iA_2$ and $B_1^* = B_1+iB_2$

The solutions can be written as,

$$x = (A_1+iA_2)e^{at} (\cos bt + i\sin bt)$$

$$y = (B+iB_2)e^{at} (\cos bt + i\sin bt)$$

(or)

$$x = e^{at}\{(A_1\cos bt - A_2\sin bt)+i(A_1\sin bt+A_2\cos bt)\}$$

$$y = e^{at}\{(B_1\cos bt - B_2\sin bt)+i(B_1\sin bt+B_2\cos bt)\}$$

$$\Rightarrow \quad x = e^{at}(A_1\cos bt-A_2\sin bt)$$

$$y = e^{at}(B_1\cos bt-B_2\sin bt)$$

and

$$x = e^{at}(A_1\sin bt+A_2\cos bt)$$

$$y = e^{at}(B_1\sin bt+B_2\cos bt)$$

These solutions are L.I.

$$x = e^{at}\{c_1(A_1\cos bt-A_2\sin bt) + c_2(A_1\sin bt+A_2\cos bt)\}$$

$$y = e^{at}\{c_1(B_1\cos bt - B_2\sin bt) + c_2(B_1\sin bt + B_2\cos bt)\}$$

Solve:

1) $\frac{dx}{dt} = -3x + 4y$

$\frac{dy}{dt} = -2x + 3y$

Solution:

Let $x = Ae^{mt}$, $y = Be^{mt}$

Sub in the equation.

$$mAe^{mt} = -3Ae^{mt} + 4Be^{mt}$$

$$mBe^{mt} = -2Ae^{mt} + 3Be^{mt}$$

$$\Rightarrow mA = -3A + 4B$$

$$mB = -2A + 3B$$

$$\left.\begin{aligned} A(m+3)-4B &= 0 \\ B(m-3)+2A &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (1)$$

$$\begin{vmatrix} m+3 & -4 \\ 2 & m-3 \end{vmatrix} = 0$$

$$(m+3)(m-3)+8 = 0$$

$$m^2-9+8 = 0$$

$$m^2-1 = 0$$

$$m^2 = 1$$

$$m = \pm 1$$

put m = 1 in (1)

$$4A-4B = 0$$

$$2A-2B = 0$$

$\therefore 4A = 4B$

$\Rightarrow A = B$

$\Rightarrow A = B = 1$

$\therefore$ The solution $x_1 = Ae^{m_1 t}$, $y_1 = Be^{m_1 t}$

$\therefore$ The solution is $x_1 = e^t$

$y_1 = e^t$

put m = -1 in (1)

$2A-4B = 0$

$2A-4B = 0$

Which reduces to A-2B = 0

Take B = 1, A = 2.

$\therefore$ The corresponding solution is

$x_2 = Ae^{m_2 t}$, $y_2 = Be^{m_2 t}$

$$\therefore \left\{\begin{array}{l} x_2 = 2e^{-t} \\ y_2 = e^{-t}, \end{array}\right.$$

The general solution is

$x = c_1x_1+c_2x_2$

$y = c_1y_1+c_2y_2$

(i.e) $x = c_1e^t+2c_2e^{-t}$

$y = c_1e^t+c_2e^{-t}$

2. Solve:

$$\frac{dx}{dt} = 3x+4y$$
$$\frac{dy}{dt} = x-y$$

Solution:

Let the solution be $x = Ae^{mt}$, $y = Be^{mt}$

Sub in the equation.

$$Ame^{mt} = 3Ae^{mt} - 4Be^{mt}$$

$$Bme^{mt} = Ae^{mt} - Be^{mt}$$

Which reduces to

$$\left.\begin{aligned} (m-3)A+4B &= 0 \\ -A+(m+1)B &= 0 \end{aligned}\right\} \quad \text{......... (1)}$$

$$\begin{vmatrix} m-3 & 4 \\ -1 & m+1 \end{vmatrix} = 0$$

$$(m-3)(m+1)+4 = 0$$

$$m^2-3m+m-3+4 = 0$$

$$m^2-2m+1 = 0$$

$$\therefore (m-1)(m-1) = 0$$

$$\therefore m = 1, 1$$

put m = 1 in (1) we get

$$-2A+4B = 0$$

$$-A+2B = 0$$

$$\Rightarrow -A+2B = 0$$

$$\Rightarrow A = 2B$$

Take B = 1, ∴ A = 2.

∴ The corresponding soln is

$$x_1 = Ae^{mt}, y = Be^{mt}$$

$$\begin{cases} x_1 = 2e^t \\ y_1 = e^t \end{cases}$$

Let us assume that, the second soln is

$$x_2 = (A_1+A_2t)e^t$$

$$y_2 = (B_1+B_2t)e^t$$

$$\frac{dx^2}{dt} = (A_1 + A_2t)e^t + A_2e^t$$

$$\frac{dy^2}{dt} = (B_1 + B_2t)e^t + B_2e^t$$

Sub in the given equation

$$\frac{dx}{dt} = 3x - 4y$$

$$\frac{dy}{dt} = x - y$$

$$(A_1+A_2t)e^t+A_2e^t = 3(A_1+A_2t)e^t - 4(B_1+B_2t)e^t$$

$$(B_1+B_2t)e^t+B_2e^t = (A_1+A_2t)e^t - (B_1+B_2t)e^t$$

Which is same as,

$$A_1+A_2t+A_2 = 3(A_1+A_2t)- 4(B_1+B_2t)$$

$$B_1+B_2t+B_2 = (A_1+A_2t)- (B_1+B_2t)$$

Equating the constant term

$$\left.\begin{aligned} A_1+A_2 &= 3A_1-4B_1 \\ B_1+B_2 &= A_1-B_1 \end{aligned}\right\} \quad \text{......... (2)}$$

Equating the coeff of 't'

$$\left.\begin{aligned} A_2 &= 3A_2-4B_2 \\ B_2 &= A_2-B_2 \end{aligned}\right\} \quad \text{......... (3)}$$

(3) $\Rightarrow A_2-2B_2 = 0$

Take $B_2 = 1$, $\therefore A_2 = 2$.

Sub in (2)

$$A_1+2 = 3A_1-4B_1$$
$$B_1+1 = A_1-B_1$$

$$\Rightarrow 2A_1-4B_1 = 2$$

$$A_1-2B_1 = 1$$

Take $B_1 = 0 \therefore A_1 = 1$

The second solution is

$$x_2 = (A_1+A_2t)e^t$$
$$= (1+2t)e^t$$
$$y_2 = (B_1+B_2t)e^t$$
$$= (0+t)e^t$$
$$= te^t$$

Hence the solutions are,

$$\begin{cases} x_1 = 2e^t \\ y_1 = e^t \end{cases} \text{ and } \quad x_2 = (3+2t)e^t, \quad y_2 = (1+t)e^t$$

The general solution is

$$x = c_1x_1 + c_2x_2$$
$$y = c_1y_1 + c_2y_2$$
$$\therefore \quad x = c_12e^t + c_2(1+2t)e^t$$
$$y = c_1e^t + c_2te^t$$

3. Solve:

$$\frac{dx}{dt} = x-2y$$

$$\frac{dy}{dt} = 4x+5y$$

Solution:

Let the solution be assumed as,

$x = Ae^{mt}$

$y = Be^{mt}$

Sub in the given equation.

$Ame^{mt} = Ae^{mt} - 2Be^{mt}$

$Bme^{mt} = 4Ae^{mt} + 5Be^{mt}$

Which reduce to

$Am = A-2B$

$Bm = 4A+5B$

$(m-1)A+2B = 0$

$4A+(5-m)B = 0$

$$\begin{vmatrix} m-1 & 2 \\ 4 & 5-m \end{vmatrix} = 0$$

$(m-1)(5-m)-8 = 0$

$5m-m^2-5+m-8 = 0$

$m^2-6m+13 = 0$

$$m = \frac{6 \pm \sqrt{36-52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm i4}{2}$$

$= 3 \pm 2i$

m 3+2i, 3-2i

Roots are complex.

$m = 3+2i$

Let the solution be $x = A^*e^{mt}$

$y = B^*e^{mt}$

$A^* = A_1+iA_2$

$B^* = B_1+iB_2$ Where A* and B* are complex no.

The soln becomes

$x = (A_1+iA_2)e^{(3+2i)t}$

$= (A_1+iA_2)e^{3t}e^{2it}$

$y = (B_1+iB_2)e^{(3+2i)t}$

$= (B_1+iB_2)e^{3t}e^{2it}$

(i.e) $x = e^{3t}(A_1+iA_2)(\cos 2t+i\sin 2t)$

$y = e^{3t}(B_1+iB_2)(\cos 2t+i\sin 2t)$

(i.e) $x = e^{3t}\{[A_1\cos 2t-A_2\sin 2t] + i[A_2\cos 2t+A_1\sin 2t]\}$

$y = e^{3t}\{[B_1\cos 2t-B_2\sin 2t] + i[B_2\cos 2t+B_1\sin 2t]\}$

We can take the solution is

$x_1 = e^{3t}(A_1\cos 2t-A_2\sin 2t)$

$y_1 = e^{3t}(B_1\cos 2t-B_2\sin 2t)$

$x_2 = e^{3t}(A_2\cos 2t-A_1\sin 2t)$

$y_2 = e^{3t}(B_2\cos 2t-B_1\sin 2t)$

Sub in the system

$$\frac{dx}{dt} = x-2y$$

$$\frac{dx_1}{dt} = x_1-2y_1$$

We have,

$x_1 = e^{3t}(A_1\cos 2t-A_2\sin 2t)$

$$\frac{dx}{dt} = 3e^{3t}(A_1\cos 2t - A_2\sin 2t + e^{3t}(-2A_1\sin 2t - 2A_2\cos 2t)$$

$$= e^{3t}\{(3A_1-2A_2)\cos 2t-(3A_2+2A_1)\sin 2t\}$$

Sub in the equation $\frac{dx}{dt} = x_1 - 2y_1$

$$e^{3t}\{(3A_1-2A_2)\cos 2t-(3A_2+2A_1)\sin 2t\} = e^{3t}(A_1\cos 2t-A_2\sin 2t)-2e^{3t}(B_1\cos 2t-B_2\sin 2t)$$

$$(3A_1-2A_2)\cos 2t-(3A_2+2A_2)\sin 2t = A_1\cos 2t-A_2\sin 2t-2B_1\cos 2t+2B_2\sin 2t$$

Equating the coeff. of cos2t

$$3A_1-2A_2 = A_1-2B_1$$

$$2A_1+2B_1-2A_2 = 0$$

$$\Rightarrow A_1+B_1-A_2 = 0 \qquad \text{......... (1)}$$

Equating the coeff of sin2t

$$-3A_2-2A_1 = -A_1+2B_2$$

$$3A_2+2A_1 = A_2-2B_2$$

$$\therefore 2A_2+2A_1+2B_2 = 0$$

$$\Rightarrow \quad A_1+A_2+B_2 = 0 \qquad \text{......... (2)}$$

Again $\frac{dy_1}{dt} = 4x_1 5y_1$

$$y_1 = e^{3t}[B_1\cos 2t-B_2\sin 2t]$$

$$\frac{dy_1}{dt} = 3e^{3t}(B_1\cos 2t-B_2\sin 2t) + e^{3t}[-2B_1\sin 2t-2B_2\cos 2t]$$

$$= e^{3t}[(3B_1-2B_2)\cos 2t-(3B_2+2B_1)\sin 2t]$$

Sub in the equation

$$\frac{dy_1}{dt} = 4x_1 5y_1$$

e^{3t} $\{(3B_1-2B_2)\cos 2t-(3B_2+2B_1)\sin 2t\} = 4\{e^{3t}(A_1\cos 2t-A_2\sin 2t)\}+5\{e^{3t}(B_1\cos 2t-B_2\sin 2t)\}$

Equating coefficient of cos2t

$3B_1-2B_2 = 4A_1+5B_1$

$4A_1+2B_1+2B_2 = 0$

$2A_1+B_1+B_2 = 0$ .... (3)

Equating the coefficient of sin2t

$-(3B_2+2B_1) = -4A_2-5B_2$

$3B_2+2B_1 = 4A_2+5B_2$

$4A_2+2B_2-2B_1 = 0$

$2A_2+B_2-B_1 = 0$ .... (4)

$A_1-B_1-A_2 = 0$ .... (1)

$A_2+A_2+B_2 = 0$ .... (2)

Take $B_1 = 0$

(1) $\Rightarrow A_1-A_2 = 0$

(2) $\Rightarrow A_2+A_1+B_2 = 0$

(3) $\Rightarrow 2A_1+B_2 = 0$

(4) $\Rightarrow 2A_2+B_2 = 0$

Since $A_1-A_2 = 0$

$\Rightarrow A_1 = A_2 = 1$

(2) $\Rightarrow 1+1+B_2 = 0$

$2+B_2 = 0$

$B_2 = -2$

$B_1 = 0$

$\therefore$ The soln is $x_1 = e^{3t}(\cos 2t-\sin 2t)$

$y_1 = e^{3t}(0\cos 2t-(-2)\sin 2t)$

$y_1 = e^{3t}(2\sin 2t)$

$x_2 = e^{3t}(\cos 2t+\sin 2t)$

$y_2 = e^{3t}(-2\cos 2t)$

The general soln is

$x = c_1x_1+c_2x_2$

$y = c_1y_1+c_2y_2$

$\therefore \quad x = c_1e^{3t}(\cos 2t-\sin 2t)+c_2e^{3t}(\cos 2t+\sin 2t)$

$y = c_1e^{3t}(2\sin 2t)-2c_2e^{3t}\cos 2t$

Problem

S.T. the condition $a_2b_1>0$ is sufficient but not necessary for the system.

$\frac{dx}{dt} = a_1x+b_1y$

$\frac{dy}{dt} = a_1x+b_2y$ to have two real value L.I solutions of the form $x = Ae^{mt}$, $y = Be^{mt}$.

Proof:

Two solutions of the form $\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$ will be real and independent iff the values of m_1 must be real and distinct.

The roots of the auxillary equation

$m^2-(a_1+b_2)m+(a_1b_2-a_2b_1) = 0$

must be real and distinct.

A > 0

(i.e) $b^2-4ac > 0$

$(a_1+b_2)^2-4(a_1b_2-a_2b_1) > 0$

$\therefore a_1^2+b_2^2+2a_1b_2-4a_1b_2+4a_2b_1 > 0$

$\therefore a_1^2+b_2^2-2a_1b_2+4a_1b_2 > 0$

$(a_1-b_2)^2+4a_2b_1 > 0$ .... (1)

Suppose $a_2b_1 > 0$, the above inequality is satisfied.

Again even when $a_2b_1<0$, the inequality is satisfied if $(a_1-b_2)^2 > -4a_2b_1$.

So $a_2b_1>0$ is only a sufficient condition for (1) to be satisfied and not a necessary condition.

Problem

S.T the Wronskian of the two solns, $x_1 = e^{at}\{A_1\cos bt - A_2\sin bt\}$, $y_1 = e^{at}\{B_1\cos bt- B_2\sin bt\}$ and $x_2 = e^{at}\{A_1\sin bt - A_2\cos bt\}$, $y_1 = e^{at}(B_1\sin bt+B_2\cos bt)$ is given by $W(t) = (A_1B_2-A_2B_1)e^{2at}$ and P.T. $A_1B_2-A_2B_1\neq 0$.

Proof:

$$W(t) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= \begin{vmatrix} e^{at}(A_1\cos bt - A_2\sin bt) & e^{at}(A_1\sin bt + A_2\cos bt) \\ e^{at}(B_1\cos bt - B_2\sin bt) & e^{at}(B_1\sin bt + B_2\cos bt) \end{vmatrix}$$

$$= e^{2at}\begin{vmatrix} A_1\cos bt - A_2\sin bt & A_1\sin bt + A_2\cos bt \\ B_1\cos bt - B_2\sin bt & B_1\sin bt + B_2\cos bt \end{vmatrix}$$

$$= e^{2at}\,[A_1B_1\sin bt\cos bt+A_1B_2\cos^2 bt-A_2B_1\sin^2 bt-A_2B_2\sin bt\cos bt-$$

$$A_1B_1\sin bt\cos bt+A_1B_2\sin^2 bt-A_2B_1\cos^2 bt+A_2B_2\sin bt\cos bt]$$

$$= e^{2at}\{A_1B_2(\cos^2 bt+\sin^2 bt) - A_2B_1(\sin^2 bt+\cos^2 bt)\}$$

$$= e^{2at}(A_1B_2-A_2B_1)$$

$$\therefore W(t) = e^{2at}(A_1B_2-A_2B_1)$$

Since the solutions are linearly independent

$$e^{2at}(A_1B_2-A_2B_1) \neq 0.$$

$\because e^{2at}\neq 0$,

$\Rightarrow A_1B_2 - A_2B_1\neq 0$

Consider the non-homogenous linear system.

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t)\\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t)\end{aligned}\right\} \qquad \text{......... (1)}$$

and the corresponding homo system

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1(t)x + b_1(t)y\\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y\end{aligned}\right\} \qquad \text{......... (2)}$$

(a) If $\left\{\begin{aligned} x &= x_1(t)\\ y &= y_1(t)\end{aligned}\right.$ and $\left\{\begin{aligned} x &= x_2(t)\\ y &= y_2(t)\end{aligned}\right.$ are L.I of (2), so that $\begin{aligned} x &= c_1x_1(t)+c_2x_2(t)\\ y &= c_1y_1(t)+c_2y_2(t)\end{aligned}$ is its general solution.

S.T. $x = v_1(t)x_1(t)+v_2(t)x_2(t)$

$y = v_2(t)y_1(t)+v_2(t)y_2(t)$ will be a particular soln of (1), if the functions, $v_1(t)$ and $v_2(t)$ satisfy the system.

$$v_1^1x_1+v_2^1x_2 = f_1$$

$$v_1^1y_1+v_2^1y_2 = f_2$$

This technique for finding particular solutions of non-homogeneous linear system is called the method of variation of parameter.

(b) Apply the method out lined in (a) to find a particular soln of the non-homo system

$$\frac{dx}{dt} = x + y - 5t + 2$$

$$\frac{dy}{dt} = 4x - 2y - 8t - 8$$

Solution:

Let us find the general soln of the homo system.

$$\left.\frac{dx}{dt} = x + y \quad\right\} \qquad \text{......... (1)}$$

$$\frac{dy}{dt} = 4x - 2y$$

Let $x = Ae^{mt}$

$y = Be^{mt}$ be a soln.

Sub in equ (1)

$$Ame^{mt} = Ae^{mt} + Be^{mt}$$

$$Bme^{mt} = 4Ae^{mt} - 2Be^{mt}$$

$$Am = A+B$$

$$Bm = 4A-2B$$

$$\left.\begin{array}{l}(m-1)A-B = 0 \\ 4A-(m+2)b = 0\end{array}\right\} \quad \ldots\ldots\ldots (2)$$

$$\begin{vmatrix} m-1 & -1 \\ 4 & -(m+2) \end{vmatrix} = 0$$

$$-(m-1)(m+2) + 4 = 0$$

$$-(m^2-m+2m-2)+4 = 0$$

$$-m^2-m+2+4 = 0$$

$$m^2+m-6 = 0$$

$$(m+3)(m-2) = 0$$

$$m = -3,2$$

put m = -3 in (2)

$$-4A-A = 0$$

$$4A+B = 0$$

Take A = 1, B = -4

∴ The soln is

$$x_1 = Ae^{m_1 t}$$

$= e^{-3t}$

$y_1 = Be^{m_1 t}$

$= -4e^{-3t}$

put m = 2 in (2)

$$\left.\begin{aligned} A-B &= 0 \\ 4A-4B &= 0 \end{aligned}\right\}$$

$\Rightarrow A-B = 0$

$\Rightarrow A = B = 1$

$\therefore$ The soln is

$x_2 = Ae^{mt}$

$= e^{2t}$

$y_2 = Be^{mt}$

$= e^{2t}$

The general soln is

$x = c_1x_1+c_2x_2$

$y = c_1y_1+c_2y_2$

(i.e) $x = c_1e^{-3t}+c_2e^{2t}$

$y = -4c_1e^{-3t}+c_2e^{2t}$

The given non-homo system,

$$\frac{dx}{dt} = x + y - 5t + 2$$

$$\frac{dy}{dt} = 4x - 2y - 8t - 8$$

$f_1(t) = -5t+2$, $f_2(t) = -8t-8$.

Let us find the particular solution by variation of parameters.

Let us assume

$x = v_1x_1+v_2x_2$

$y = v_1y_1+v_2y_2$ is a particular solution, where v_1, v_2 are functions of 't'.

$$x^1 = (v_1x_1^1 + v_2x_2^1) + (v_1^1x_1 + v_2^1x^2)$$

$$y^1 = (v_1y_1^1 + v_2y_2^1) + (v_1^1y_1 + v_2^1y^2)$$

To find v_1 and v_2

Take, $v_1^1x_1+v_2^1x_2 = f_1$ .... (4)

$v_1^1y_1+v_2^1y_2 = f_2$ .... (5)

Solving (4) and (5)

$(4)\times y_2 \Rightarrow v_1^1x_1y_2+v_2^1x_2y_2 = f_1y_2$

$(5)\times x_2 \Rightarrow v_1^1y_1x_2+v_2^1x_2y_2 = f_2x_2$

$v_1^1(x_1y_2-y_1x_2) = f_1y_2 - f_2x_2$

$$v_1^1 = \frac{f_1y_2 - f_2x_2}{x_1y_2 - y_1x_2}$$

From (4)

$$v_2^1x^2 = f_1 - v_1^1x_1$$

$$= f_1 - \left(\frac{f_1y_2 - f_2x_2}{x_1y_2 - y_1x_2}\right)x_1$$

$$= \frac{f_1x_1y_2 - f_1y_1x_2 - f_1x_1y_2 + f_2x_1x_2}{x_1y_2 - y_1x_2}$$

$$= \frac{f_2x_2x_1 - f_1x_2y_1}{x_1y_2 - y_1x_2}$$

$$v_2^1x^2 = \frac{x_2(f_2x_1 - f_1y_1)}{x_1y_2 - y_1x_2}$$

$$v_2^1 = \frac{f_2x_1 - f_1y_1}{x_1y_2 - y_1x_2}$$

For the given equation

$f_1 = -5t+2$, $f_2 = -8t-8$

$$\begin{cases} x = e^{-3t} \\ y = -4e^{-3t} \end{cases} \text{ and } \begin{cases} x_2 = e^{2t} \\ y_2 = e^{2t} \end{cases}$$

$$\therefore x_1y_2 - y_1x_2 = e^{-3t}e^{2t} + 4e^{-3t}e^{2t}$$

$$= e^{-t} + 4e^{-t}$$

$$= 5e^{-t}$$

$$v_1^1 = \frac{f_1y_2 - f_2x_2}{x_1y_2 - y_1x_2}$$

$$= \frac{(-5t+2)e^{2t} - (-8t-8)e^{2t}}{5e^{-t}}$$

$$= \frac{e^{3t}}{5}(-5t+8t+2+8)$$

$$v_1^1 = \frac{e^{3t}}{5}(3t+10)$$

$$v_1 = \frac{1}{5}\int e^{3t}(3t+10)dt$$

$$= \frac{1}{5}\left\{(3t+10)\frac{e^{3t}}{3} - \int \frac{e^{3t}}{3}3.dt\right\}$$

$$= \frac{1}{5}\left((3t+10)\frac{e^{3t}}{3} - \frac{e^{3t}}{3}\right)$$

$$= \frac{e^{3t}}{15}(3t+10-1)$$

$$= \frac{e^{3t}}{15}(3t+9)$$

$$= \frac{e^{3t}}{15} 3(t+3)$$

$$v_1 = \frac{e^{3t}}{15}(t+3)$$

$$v_2^1 = \frac{f_2 x_1 - f_1 y_1}{x_1 y_2 - x_2 y_1}$$

$$= \frac{(-8t-8)e^{-3t} - (-5t+2)(-4e^{-3t})}{5e^{-t}}$$

$$= \frac{e^{-2t}}{5}\left(-8t-8-20t+8\right)$$

$$= \frac{e^{-2t}}{5}\left(-28t\right)$$

$$v_2 = \frac{-28}{5}\int te^{-2t}.dt$$

$$= \frac{-28}{5}\left\{\frac{te^{-2t}}{-2} - \int \frac{e^{-2t}}{-2}dt\right\}$$

$$= \frac{-28}{5}\left\{\frac{te^{-2t}}{-2} + \frac{e^{-2t}}{-4}\right\}$$

$$= \frac{-28}{5\times -2}e^{-2t}\left\{t + \frac{1}{2}\right\}$$

$$= \frac{28}{5\times 2}e^{-2t}\left\{\frac{2t+1}{2}\right\}$$

$$v_2 = \frac{7}{5}e^{-2t}(2t+1)$$

∴ The particular solution is

$$x = v_1 x_1 + v_2 x_2$$

$$x = \frac{e^{3t}}{5}(t+3)e^{-3t} + \frac{7}{5}e^{-2t}(2t+1)e^{2t}$$

$$= \frac{1}{5}(t+3) + \frac{7}{5}(2t+1)$$

$= \frac{1}{5}(t+3)+14t+7)$

$= \frac{1}{5}(15t+10)$

$= \frac{1}{5}5(3t+2)$

x = 3t+2.

y = $v_1y_1+v_2y_2$

$= \frac{e^{3t}}{5}(t+3)(-4e^{-3t})+\frac{7}{5}e^{-2t}(2t+1)e^{2t}$

$= \frac{-4}{5}(t+3)+\frac{7}{5}(2t+1)$

$= \frac{1}{5}(-4t-12+14t+7$

$= \frac{1}{5}(10t-5)$

$= \frac{1}{5}5(2t-1)$

y = 2t-1.

∴ The required particular solution is

x = 3t+2

y = 2t-1.

Partial Differential Equations of the first order

We obtain a relation between the derivatives of the kind

$$F\left(\frac{\partial\theta}{\partial\theta},......,\frac{\partial^2\theta}{\partial x^2},......,\frac{\partial^2\theta}{\partial x\partial t},.....\right) = 0$$

Such an equation relating partial derivatives are called a partial differential Equation.

The order of a partial differential equation to be the order of the derivatives of highest order occurring in the equation. If for example, we take θ to be the dependent variable and *x*, y and t to be independent variables, then the equation $\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t}$ is a second order equation in two variables.

The equation, $\left(\frac{\partial\theta}{\partial x}\right)^3+\frac{\partial\theta}{\partial t}=0$ is a first order equation in two variables, and $x\frac{\partial\theta}{\partial x}+y\frac{\partial\theta}{\partial y}+\frac{\partial\theta}{\partial t}=0$ is a first order equation in three variables.

The are two independent variables *x* and y and z is the dependent variables, then we write $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$.

This equation can be written in the form

$$f(x, y, z, p, q) = 0$$

Formation of partial differential equation by eliminating arbitrary constant
Problem

Find the partial differential equation by eliminating the constants a and c from the equation $x^2+y^2(z-c)^2 = a^2$.

Solution:

$$x^2+y^2(z-c)^2 = a^2$$

Diff w.r.to *x*

$$2x+2(z-c)\frac{\partial z}{\partial x} = 0$$

$$x+(z-c)p = 0 \qquad \text{......(1)}$$

Diff w.r.to y

$$2y+2(z-c)\frac{\partial z}{\partial y} = 0$$

$$y+(z-c)q = 0 \qquad \text{......(2)}$$

From (1) $z-c = \dfrac{-x}{p}$

From (2) $z-c = \dfrac{-y}{q}$

$$\therefore \frac{-x}{p} = \frac{-y}{q}$$

$$\Rightarrow \frac{x}{p} = \frac{y}{q}$$

$$qx - py = o$$

Problem

Form the partial differential equation by eliminating the constants a and c from the equation $x^2+y^2=(z-c)^2\tan^2\alpha$.

Solution:

Given equation is

$$x^2+y^2 = (z-c)^2\tan^2\alpha$$

Diff w.r.to x

$$2x = 2(z-c)\frac{\partial z}{\partial x}\tan^2\alpha$$

$$\Rightarrow x = (z-c)\, p \tan^2\alpha$$

$$\Rightarrow (z-c)\tan^2\alpha = \frac{x}{p} \qquad \text{......(1)}$$

Diff w.r.to y

$$2y = 2(z-c)\frac{\partial z}{\partial y}\tan^2\alpha$$

$$y = (z-c)q+\tan^2\alpha$$

$(z-c)\tan^2\alpha = \frac{y}{q}$ ……(2)

From (1) and (2)

$$\frac{x}{p} = \frac{y}{q}$$

$\Rightarrow qx - py = 0.$

Problem

Find the partial differential equation of $f(x^2+y^2) = z$.

Solution:

Given $f(x^2+y^2) = z$

Diff w.r.to x

$$f'(x^2+y^2)2x = \frac{\partial z}{\partial x}$$

$f'(x^2+y^2)2x = p$ ……(1)

Diff w.r.to y

$$f'(x^2+y^2)2y = \frac{\partial z}{\partial y}$$

$f'(x^2+y^2)2y = q$ ……(2)

$$\frac{(1)}{(2)} \Rightarrow \frac{x}{y} = \frac{p}{q}$$

$\Rightarrow qx - py = 0$

Problem

Eliminate the arbitrary function f from $z = xy + f(x^2+y^2)$

Solution:

Given $z = xy + f(x^2+y^2)$

Diff w.r.to x

$$\frac{\partial z}{\partial x} = y + f'(x^2+y^2).2x$$

$\therefore$ p = $y+f'(x^2+y^2).2x$

p-y = $f'(x^2+y^2).2x$

$\therefore$ $f'(x^2+y^2) = \dfrac{p-y}{2x}$ ……(1)

Diff w.r.to y

$\dfrac{\partial z}{\partial y} = x+f'(x^2+y^2)2y$

q = $x+f'(x^2+y^2)2y$

$f'(x^2+y^2) = \dfrac{q-x}{2y}$ ……(2)

From (1) and (2)

$\dfrac{p-y}{2x} = \dfrac{q-x}{2y}$

$y(p-y) = x(q-x).$

Problem

Eliminate the arbitrary function from the equation z = $f\left(\dfrac{xy}{z}\right)$

Solution:

Given z = $f\left(\dfrac{xy}{z}\right)$

Para. Diff w.r.t x.

$\dfrac{\partial z}{\partial x} = f'\left(\dfrac{xy}{z}\right)\left\{\dfrac{y}{z}-\dfrac{xy}{z^2}\dfrac{\partial z}{\partial x}\right\}$

p = $f'\left(\dfrac{xy}{z}\right)\left(\dfrac{y}{z}-\dfrac{xy}{z^2}p\right)$ …… (1)

|||rly Diff w.r.to y

$\dfrac{\partial z}{\partial y} = f'\left(\dfrac{xy}{z}\right)\left(\dfrac{x}{z}-\dfrac{xy}{z^2}q\right)$

$$q = f'\left(\frac{xy}{z}\right)\left(\frac{x}{z}-\frac{xy}{z^2}q\right) \quad \ldots\ldots (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{p}{q} = \frac{\frac{y}{z}-\frac{xy}{z^2}p}{\frac{x}{z}-\frac{xy}{z^2}q}$$

$$\frac{p}{q} = \frac{y^z - xyp}{x^z - xyq}$$

$$\frac{p}{q} = \frac{y(z-xp)}{x(z-yq)}$$

x(z-yq)p = y(z-xp)q

$x$zp - $x$ypq = yzq - $x$ypq

$\Rightarrow x$zp = yzp

$\Rightarrow x$p = yq

$\Rightarrow x$p - yq = 0

Problem

Eliminate arbitrary function from $f(x^2+y^2+z^2, z^2-2xy) = 0$.

Solution:

Given equation $f(x^2+y^2+z^2, z^2-2xy) = 0$.

Which may be taken as,

$$z^2-2xy = g(x^2+y^2+z^2)$$

Diff w.r.to x

$$2z\frac{\partial z}{\partial x} - 2y = g'\left(x^2+y^2+z^2\right)\left(2x+2z\frac{\partial z}{\partial x}\right)$$

$$2(zp-y) = g'\left(x^2+y^2+z^2\right)2(x+zp) \quad \ldots\ldots (1)$$

Diff w.r.to y

$$2z\frac{\partial z}{\partial y} - 2x = g'\left(x^2+y^2+z^2\right)\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$2zq - 2x = g'\left(x^2+y^2+z^2\right)2(y+zq)$$

$$zq - x = g'\left(x^2+y^2+z^2\right)(y+zq) \qquad \ldots\ldots (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{zp-y}{zq-x} = \frac{x+zp}{y+zq}$$

(zp-y) (y+zp) = (x+zp) (zq-x)

Cauchy's Problem for First - order equations

(a) $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ are functions, which together with their first derivatives, are continuous in the interval M defined by $\mu_1 < \mu < \mu_2$.

(b) And if F(x, y, z, p, q) is a continuous function of x, y, z, p and q in a certain region U of the $x$yzpq space, then it is required to establish the existence of a function $\varphi(x, y)$ with the following properties.

(1) $\varphi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

(2) For all values of x and lying in R_1 the point $\{x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y)\}$ lies in U and $F[x, y, \varphi(x, y), \varphi_x(x, y), \varphi_y(x, y)] = 0$

(3) For all μ belonging to the interval M the point $\{x_0(\mu), y_0(\mu)\}$ belonging to the region and $\varphi(x_0(\mu), y_0(\mu)) = z_0$

Linear Equations of the First order

A first order linear partial differential equation of the form.

$$Pp + Qq = R$$

Where P, Q, R are functions of x, y, z is called Lagrange's equation,

Theorem:

The general solution of the linear partial differential equation Pp+Qq = R is F(u,v) = 0, where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Proof:

Given u(*x*, y, z) = c_1, v(*x*, y, z) = c_2 is a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ …… (1)

From, u(*x*, y, z) = c_1 we get,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \qquad \text{…… (2)}$$

From (1) and (2) we find

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0 \qquad \text{…… (3)}$$

|||rly, consider v(*x*, y, z) = c_2 and equ (1) we get,

$$P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z} = 0 \qquad \text{…… (4)}$$

From (3) and (4)

$$\frac{p}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{Q}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{R}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

(i.e) $$\frac{p}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \qquad \text{…… (5)}$$

We know, F(u, v) = 0 is the general soln of partial differential equation,

$$\frac{\partial(u,v)}{\partial(y,z)}p + \frac{\partial(u,v)}{\partial(z,x)}q = \frac{\partial(u,v)}{\partial(x,y)} \qquad \text{…… (6)}$$

From (5) and (6)

Pp + Qq = R

Hence u(*x*, y, z) = c_1 and v(*x*, y, z) = c_2 is a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Thus F(u, v) = 0 is a general soln of

Pp+ Qq = R

The result in the above theorem can be extended to any number of variables.

The general soln of

$$P_1\frac{\partial z}{\partial x_1}+P_2\frac{\partial z}{\partial x_2}+....+P_n\frac{\partial z}{\partial x_n}= \quad R \text{ is } F(u_1, u_2,\ldots,u_n) \quad = \quad 0$$

Where $u_1(x_1, x_2,\ldots,x_n) = c_1$, $u_2(x_1,x_2,\ldots,x_n) = c_{2\ldots\ldots}$ $u_n(x_1,x_2,\ldots,x_n) = c_n$ is a solution of

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=.....=\frac{dx_n}{P_n}=\frac{dz}{R}$$

Problem

Find the general solution of the differential equation, $x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z$

Solution:

Given $x^2p+y^2q = \quad (x+y)z$

$P = x^2$, $Q = y^2$, $R = (x+y)z$

The auxillary equ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

Take, $\frac{dx}{x^2} = \frac{dy}{y^2}$

$$\int\frac{dx}{x^2} = \int\frac{dy}{y^2}$$

$$\frac{-1}{x} = \frac{-1}{y}-c$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{y}+c$$

$$\Rightarrow \quad \frac{1}{x}-\frac{1}{y} = c_1$$

Take $\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$

$$\frac{dx - dy}{x - y} = \frac{dz}{z}$$

$\log(x\text{-y}) = \log z + \log c_2$

$\log(x\text{-y}) = \log zc_2$

$\Rightarrow x\text{-y} = zc_2$

$$\Rightarrow \frac{x - y}{z} = c_2$$

The general soln of the given equ is F(u, v) = 0

$$\therefore F\left(\frac{1}{x} - \frac{1}{y}, \frac{x - y}{z}\right) = 0$$

$$\text{(i.e)} \frac{x - y}{z} = f\left(\frac{1}{x} - \frac{1}{y}\right)$$

Problem

Find the general soln of the equ $z(xp\text{-yq}) = y^2\text{-}x^2$

Solution:

Given equ is $z(xp - yq) = y^2 - x^2$

$zxp - zyq = y^2 - x^2$

$Pp + Qq = R$

$\therefore P = zx, Q = -zy, R = y^2 - x^2$

Auxillary equ is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}$$

Take $\frac{dx}{zx} = \frac{dy}{-zy}$

$$\frac{dx}{x} = \frac{-dy}{y}$$

$\int$ ing

$$\log x = -\log y + \log c_1$$

$$\log x + \log y = \log c_1$$

$$\Rightarrow \log xy = \log c_1$$

$$xy = c_1$$

Again

$$\frac{dx+dy}{zx-yz} = \frac{dy}{y^2-x^2}$$

$$\frac{dx+dy}{z(x-y)} = \frac{-dz}{(x-y)(x+y)}$$

$$\Rightarrow \frac{dx+dy}{z} = -\frac{dz}{x+y}$$

$$(x+y)(dx+dy) = -zdz$$

$$(x+y)d(x+y) = -zdz$$

$\int$ ing

$$\frac{(x+y)^2}{2} = \frac{-z^2}{2} + \frac{c^2}{2}$$

$$(x+y)^2+z^2 = c_2$$

The general soln is given bu F(u, v) = 0

(i.e) $F(xy, (x+y)^2+z^2) = 0$ or $v = f(u)$

$$(x+y)^2+z^2 = f(xy).$$

Problem

If u is a function of x, y and z which satisfies the partial differential equation.

Solution:

$$(y-z)\frac{\partial u}{\partial x}+(z-x)\frac{\partial u}{\partial y}+(x-y)\frac{\partial u}{\partial z}=0$$

Show that u contains x, y and z only in combinations $x+y+z$ and $x^2+y^2+z^2$

The auxillary equ is

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}=\frac{du}{0}$$

∴ We get, du = 0 ∴ $u = c_1$

$$\frac{dx+dy+dz}{y-z+z-x+x-y} = 0$$

$$\Rightarrow \quad dx+dy+dz = 0$$

$$\Rightarrow x+y+z = c_2$$

Again $\frac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)} = \frac{du}{0}$

$$\frac{xdx+ydy+zdz}{xy-zx+yz-xy+zx-zy} = \frac{du}{0}$$

$$\Rightarrow xdx+ydy+zdz = 0$$

$$\therefore \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \frac{c_3}{2}$$

$$\therefore \quad x^2+y^2+z^2 = c^3$$

If u is the soln, $u(c_1,c_2,c_3) = 0$

(i.e)$u(c_1, x+y+z, x^2+y^2+z^2) = 0$

$$u = f(x+y+z, x^2+y^2+z^2).$$

Nonlinear Partial Differential Equation of the first order.

The solutions of the partial differential equation of the first order will contain two constants and may be in the form, $F(x,y,z,a,b) = 0$. In this case the solution is said to be the complete solution or complete integral.

A solution of the partial differential equation will be in terms of two arbitrary functions in the form F(u, v) = 0, In this case the solution is called general solution or general integral.

The complete solution of the partial differential equation of the first order is of the form $F(x,y,z,a,b) = 0$, where a and b are arbitrary constants.

Consider, this solution as a function of a, b

$$\text{(i.e)} \quad \varphi(a, b) = 0 \qquad \ldots\ldots (1)$$

Take

$$\frac{\partial \varphi}{\partial a} = 0 \qquad \ldots\ldots (2)$$

$$\frac{\partial \varphi}{\partial b} = 0 \qquad \ldots\ldots (3)$$

The equation obtained by eliminating a and b from (1), (2), (3) is known as the general singular solution of the differential Equation.

Envelope

Consider the complete solution of partial differential equation of the form $\varphi(a, b) = 0$, If we can express one the constants in terms of the other say $b = f(a)$ then

$$\varphi(a,f(a)) = 0 \qquad \ldots\ldots (1)$$

$$\frac{\partial \varphi}{\partial a} = 0 \qquad \ldots\ldots (2)$$

Eliminating 'a' from (1) and (2) we get the envelope of the family of surfaces which are solution of the given differential equation.

Problem

Verify that $z = ax+by+a+b-ab$ is a complete integral of the partial differential equation $z = px+qy+p+q-pq$, where a and b are constants. Show that the envelope of all planes corresponding to complete integrals provides a singular solution of the differential equation, and determine a general solution by finding the envelope of those planes that pass through the origin.

Solution:

Given $z = ax+by+a+b-ab \qquad \ldots\ldots (1)$

$$\frac{\partial z}{\partial x} = a \qquad a = p$$

$$\frac{\partial z}{\partial y} = b \qquad b = q$$

Eliminating a and b

We get, z = px+qy+p+q-pq …… (2)

∴ (1) is a complete solution of (2)

Let P(a, b) = ax+by+a+b-ab-z …… (3)

$$\frac{\partial \varphi}{\partial a} = x+1-b$$

$$\frac{\partial \varphi}{\partial b} = y+1-a$$

$$\frac{\partial \varphi}{\partial a} = 0 \Rightarrow x+1-b = 0$$

$$b = x+1$$

$$\frac{\partial \varphi}{\partial b} = 0 \Rightarrow y+1-a = 0$$

$$a = y+1$$

φ(a, b) = 0

$$\therefore z = ax+by+a+b-ab$$

$$= a(1+x)+b(y+1)-ab$$

$$= (y+1)(x+1)+(x+1)(y+1)-(x+1)(y+1)$$

$$= (x+1)(y+1)$$

∴ The general singular solution is

$$z = (x+1)(y+1)$$

The given plane passes through the origin

$$z = ax+by+a+b-ab$$

$$\Rightarrow 0 = 0+0+a+b-ab$$

a+b-ab = 0

b(1-a) = -a

$\therefore$ b = $\frac{-a}{1-a}$

= $\frac{a}{a-1}$

We have,

z = $ax+\left(\frac{a}{a-1}\right)y$

f(x,y,z,a,φ(a)) = $ax+\left(\frac{a}{a-1}\right)y-z$

Eliminating 'a' from (3) and $\frac{\partial f}{\partial a}=0$, we get the required solution.

Problem

Verify that the equations

(a) $z = \sqrt{2x+a}+\sqrt{2y+b}$

(b) $z^2+\mu = 2(1+\lambda^{-1})(x+\lambda y)$ are both complete integrals of the partial differential equation

Solution:

$$z = \frac{1}{p}+\frac{1}{q}$$

We have, $z = \sqrt{2x+a}+\sqrt{2y+b}$

$$\therefore \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{2x+a}}2+0$$

$$\therefore P = \frac{1}{2\sqrt{2x+a}}$$

$$\therefore \sqrt{2x+a} = \frac{1}{p}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{2y+b}}2$$

$$q = \frac{1}{\sqrt{2y+b}}$$

$$\therefore \frac{1}{\sqrt{2y+b}} = \frac{1}{q}$$

We have $z = \sqrt{2x+a} + \sqrt{2y+b}$

$$\therefore z = \frac{1}{p} + \frac{1}{q}$$

Again $z^2+\mu = 2(1+\lambda^{-1})(x+\lambda y)$

$$2z\frac{\partial z}{\partial x} = 2(1+\lambda^{-1})$$

$$zp = (1+\lambda^{-1}) \qquad \ldots\ldots (1)$$

Also, $2z\frac{\partial z}{\partial x} = 2(1+\lambda^{-1})\lambda$

$$zq = \lambda(1+\lambda^{-1}) \qquad \ldots\ldots (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{p}{q} = \frac{1}{\lambda}$$

$$\Rightarrow \lambda^{-1} = \frac{p}{q}$$

$$(1) \Rightarrow zp = 1 + \frac{p}{q}$$

$$z = \frac{1}{p} + \frac{p}{qp}$$

$$z = \frac{1}{p} + \frac{1}{q}$$

$\therefore z = \sqrt{2x+a} + \sqrt{2y+b}$ and $z^2+\mu = 2(1+\lambda^{-1})(x+\lambda y)$ are both complete of the partial differential equation $z = \frac{1}{p} + \frac{1}{q}$.

Problem

Compatible system of first order equation

Consider the first order Partial differential equation is

$$f(x, y, z, p, q) = 0 \qquad \ldots\ldots (1)$$

$$g(x, y, z, p, q) = 0 \qquad \ldots\ldots (2)$$

If every solution of (1) is a solution of (2) and every solution of (2) is a solution of (1). Then (1) and (2) are called compatible.

Definition:

Two equations are said to compatible if every solution of one is a solution of the another.

To find the condition that two P.D.E of first order are compatible.

Let the given equation be

$$f(x, y, z, p, q) = 0 \qquad \ldots\ldots (1)$$

and $$g(x, y, z, p, q) = 0 \qquad \ldots\ldots (2)$$

The equation can be solved for p and q if

$$J = \frac{\partial(f, g)}{\partial(p, q)} \neq 0$$

If the equations are compatible we must be able to solve for p and q.

$$\therefore J \neq 0$$

Let $p = \varphi(x, y, z)$ and $q = \psi(x, y, z)$

The solution of the diff. equation can be obtained for $dz = pdx+qdy$ which is integrable

$$\Rightarrow pdx+qdy-dx = 0 \qquad \ldots\ldots (3)$$

$$\Rightarrow \varphi dx+\psi dy-dz = 0$$

Take $\bar{x} = (\varphi, \psi, -1)$

$$\text{curl } \bar{x} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi & \psi & -1 \end{vmatrix}$$

$$= \vec{i}\left(0-\frac{\partial\psi}{\partial z}\right)-\vec{j}\left(0-\frac{\partial\varphi}{\partial z}\right)+\vec{k}\left(\frac{\partial\psi}{\partial y}-\frac{\partial\varphi}{\partial y}\right)$$

$$\text{curl } \bar{x} = -\vec{i}\frac{\partial\psi}{\partial z}+\vec{j}\frac{\partial\varphi}{\partial z}+\vec{k}\left(\frac{\partial\psi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)$$

$$= -\vec{i}\,\psi_z+\vec{j}\,\varphi_z+\vec{k}(\psi_x-\varphi_y)$$

$\bar{x}$ curl $\bar{x} = 0$

$(\varphi,\psi,-1)\ (-\psi,\varphi_z,\psi_z-\varphi_y) = 0$

$-\varphi\psi_z+\psi\varphi_z-\psi_x+\varphi_y = 0$

$\varphi_y+\psi\varphi_z = \psi_x+\varphi\psi_z$ …… (4)

$f(x,y,z,p,q) = 0$

Diff. w.r.to x

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad p = \varphi,\ q = \psi$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial \varphi}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial \psi}{\partial x} = 0$$

|||rly diff w.r.to z

$$\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\frac{\partial \varphi}{\partial z}+\frac{\partial f}{\partial q}\frac{\partial \psi}{\partial z} = 0$$

∴ we get,

$f_x+f_p\varphi_x+f_q\psi_x = 0$ …… (5)

$f_z+f_p\varphi_z+f_q\psi_z = 0$ …… (6)

(6) ×φ $\varphi f_z+\varphi f_p\varphi_z+\varphi f_q\psi_z = 0$ …… (7)

(5) + (7) ⇒ $f_x+\varphi f_z+f_p[\varphi_x+\varphi\varphi_z)+f_q(\psi_x+\varphi\psi_z) = 0$ …… (8)

|||rly for the equation

$g(x,y,z,p,q) = 0$

We get

$$g_x+\varphi y_z+g_p[\varphi_x+\varphi\varphi_z]+g_q(\psi_x+\varphi\psi_z) = 0 \quad \ldots\ldots (9)$$

Take (8) × g_p

$$f_x g_p+\varphi f_z g_p+g_p f_p[\varphi_x+\varphi\varphi_z]+f_q g_p(\psi_x+\varphi\psi_2) = 0 \quad \ldots\ldots (10)$$

(9) × f_p

(i.e) $g_x f_p+\varphi g_z f_p+f_p g_p[\varphi_x+\varphi\varphi_z]+f_p g_q(\psi_x+\varphi\psi_z) = 0$ …… (11)

(10) - (11)

$$[f_x g_p-g_x f_p]+\varphi[f_z g_p-f_p g_z]+0(f_q g_p-f_p g_q)(\psi_x+\varphi\psi_z) = 0$$

$$\frac{\partial(f,g)}{\partial(x,p)}+\varphi\frac{\partial(f,g)}{\partial(z,q)}-\frac{\partial(f,g)}{\partial(p,q)}(\psi_x+\varphi\psi_z) = 0$$

$$\frac{\partial(f,g)}{\partial(p,q)}(\psi_x+\varphi\psi_z) = \frac{\partial(f,g)}{\partial(x,p)}+\varphi\frac{\partial(f,g)}{\partial(z,q)}$$

$$J(\psi_x+\varphi\psi_z) = \frac{\partial(f,g)}{\partial(x,p)}+\varphi\frac{\partial(f,g)}{\partial(z,q)}$$

$$\therefore \psi_x+\varphi\psi_z = \frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)}+\varphi\frac{\partial(f,g)}{\partial(z,q)}\right] \quad \ldots\ldots (I)$$

|||^{rly} diff (1) and (2) w.r.to y and z

We get

$$\psi_y+\psi\varphi_z = \frac{-1}{J}\left[\frac{\partial(f,g)}{\partial(y,q)}+\psi\frac{\partial(f,g)}{\partial(z,q)}\right] \quad \ldots\ldots (II)$$

But by (4)

$$\psi_x+\varphi\psi_z = \varphi_y+\psi\varphi_z$$

Using this in I and II we get

$$\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,p)}+\varphi\frac{\partial(f,g)}{\partial(z,q)}\right] = \frac{-1}{J}\left[\frac{\partial(f,g)}{\partial(y,p)}+\psi\frac{\partial(f,g)}{\partial(z,q)}\right]$$

$$\frac{\partial(f,g)}{\partial(x,p)}+\varphi\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+\psi\frac{\partial(f,g)}{\partial(z,q)} = 0$$

Since φ = p, ψ = q

$$\therefore \frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

This is the required condition for the equation to be compatible.

We can write as [f, g] = 0.

Problem

Show that the equation xp-yq = 0, z(xp+yq) = 2xy are compatible and solve theorem.

Solution:

Given f	=	$xp-yq$	g	=	$z(xp+yq)-2xy$
f_x	=	p	g	=	$zxp+zyq-2xy$
f_y	=	-q	g_x	=	$zp-2y$
f_z	=	0	g_y	=	$zq-2x$
f_p	=	x	g_z	=	$xp+yq$
f_q	=	-y	g_p	=	zx
			g_q	=	zy

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix}$$

$$= \begin{vmatrix} f_x & f_p \\ g_x & y_p \end{vmatrix}$$

$$= \begin{vmatrix} p & x \\ zp-2y & zx \end{vmatrix}$$

$= pzx-x(zp-2y)$

$= pzx-pzx+2yx$

$= 2yx$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x \\ xp+yq & zx \end{vmatrix}$$

$$= -x^2p-xyq$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix}$$

$$= -zyq+y(zq-2x)$$

$$= -zyq+yzq-2xy$$

$$= -2xy$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -y \\ xq-yq & zy \end{vmatrix}$$

$$= 0+y(xp+yq)$$

$$= xyp+y^2q$$

We have,

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$$= 2xy+p(-x^2p-xyq)-2xy+q(xyp+y^2q)$$

$$= 2xy-x^2p^2-xypq-2xy+xypq+y^2q^2$$

$$= y^2q^2-x^2p^2$$

$$= p^2x^2-x^2p^2 \qquad \because px - qy = 0$$

$= \quad 0 \qquad \Rightarrow px = qy$

$\therefore [f, g] = 0.$

$\therefore$ The two equations are compatible.

Let us find p and q. From the given equation.

$$xp - yq = 0 \qquad \ldots\ldots (1)$$

$$z(xp + yq) = 2xy \qquad \ldots\ldots (2)$$

(1) $\Rightarrow xp = yq$

$$\therefore (2) \Rightarrow z(yq + yq) = 2xy$$

$$z2yq = 2xy$$

$$\therefore zq = x$$

$$\therefore q = \frac{x}{z}$$

$$xp = yq$$

$$xp = y\frac{x}{z}$$

$$= \frac{y}{z}$$

Solution is given by

$$dz = pdx + q.dy$$

$$= \frac{y}{z}dx + \frac{x}{z}dy$$

$$\therefore zdz = ydx + xdy$$

$$\therefore zdz = d(xy)$$

$$\int zdz = \int d(xy)$$

$$\frac{z^2}{z} = xy + \frac{c}{2}$$

$$\Rightarrow \frac{z^2}{z} = \frac{2xy}{2} + \frac{c}{2}$$

$$\Rightarrow z^2 = 2xy + c$$

Problem

Such that the equation $xp+yq = x$ and $x^2p+q = xz$ are compatible and find the solution.

Solution:

Let $f = xp-uq-x,\quad g = x^2p+q-xz$

$f_x = p-1 \quad g_x = 2xp-z$

$f_y = -q \quad g_y = 0$

$f_z = 0 \quad g_z = -x$

$f_p = x \quad g_p = x^2$

$f_q = -y \quad g_q = 1$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$$

$$= \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix}$$

$$= (p-1)x^2 - x(2xp-z)$$

$$= px^2 - x^2 - 2x^2p + xz$$

$$= -x^2 - x^2p + xz$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix}$$

$$= 0 + x^2 = x^2$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} f_z & f_q \\ f_z & g_q \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy$$

We have,

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$= -x^2-x^2p+xz+px^2-q+q(-xy)$

$= -x^2-x^2p+xz+px^2-q-xyq$

$= -x^2+xz-q-xyq$

$= -x^2+x^2p+q-q-qxy$ [from (2)]

$= -x^2+x^2p-xyq$ $x^2 = x^2p+q$

$= x^2-x^2p+xyq$

$= x[x-xp+qy]$

$= x(0)$

$\therefore [f, g] = 0.$ [from (1)]

$\therefore$ The equations are compatible.

Let us find p and q from the gn equ.

$xp-yq = 0$ …… (1)

$x^2p+q = xz$ …… (2)

From (2) $q = xz-x^2p$

Sub in (1)

$xp-y(xz-x^2p) = x$

$xp - xyz + x^2yp = x$

$p(x+x^2y) = x+xyz$

$px(1+xy) = x(1+yz)$

$$p = \frac{1+yz}{1+xy}$$

$$q = xz - x^2\left[\frac{1+yz}{1+xy}\right]$$

$$= \frac{xz(1+xy)-x^2-x^2yz}{1+xy}$$

$$= \frac{xz+x^2yz-x^2-x^2yz}{1+xy}$$

$$= \frac{xz-x^2}{1+xy}$$

$$q = \frac{x(z-x)}{1+xy}$$

The soln is obtained from the equation

$$dz = pdx+qdy$$

$$dz = \frac{1+y^2}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$$= \frac{(1+xy)+(yz-xy)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$$= dx + \frac{yz-xy}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$$\therefore dz-dx = \frac{y(z-x)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$$= (z-x)\left\{\frac{y}{1+xy}dx + \frac{x}{1+yx}dy\right\}$$

$$\Rightarrow \frac{dz-dx}{z-x} = \frac{ydx+xdy}{1+xy}$$

$$\int \frac{dz-dx}{z-x} = \int \frac{ydx+xdy}{1+xy}$$

$$\Rightarrow \log(z-x) = \log(1+xy)+\log c$$

$$\Rightarrow z-x = c(1+xy)$$

$$\Rightarrow z = x+c(1+xy)$$

Problem

Show that The equation f(*x*,y,p,q) = 0, g(*x*,y,p,q) = o are compatible if $\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$ verify that the equation p = P(*x*,y), q = Q(*x*,y) are compatible if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

Solution:

f(*x*,y,p,q) = 0, and g(*x*,y,p,q) = 0

$\therefore f_z = 0 \qquad g_z = 0$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}$$

$$= \begin{vmatrix} 0 & f_p \\ 0 & g_p \end{vmatrix} = 0$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= \begin{vmatrix} 0 & f_q \\ 0 & g_g \end{vmatrix} = 0$$

The given equation are compatible if [f, g] = 0

$$\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)}+0+\frac{\partial(f,g)}{\partial(y,q)}+0 = 0$$

Hence $\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)} = 0$

Consider, p = P(*x*, y), q = Q(*x*, y)

∴ f	=	p-P(x, y)	q	=	q-Q(x, y)
f_x	=	$-P_x$	g_x	=	$-Q_x$
f_y	=	$-P_y$	g_x	=	$-Q_y$
f_z	=	0	g_z	=	0
f_p	=	1	g_p	=	0
f_q	=	0	g_q	=	1

Since both f and g are free from z, we have as in the above the equ are compatible if

$$\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)} = 0 \qquad \ldots\ldots (1)$$

Now, $\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$

$$= \begin{vmatrix} -P_x & 1 \\ -Q_x & 0 \end{vmatrix}$$

$$= Q_x$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} -P_y & 1 \\ -Q_y & 0 \end{vmatrix}$$

$$= -P_y$$

∴ (1) ⇒ $Q_x - P_y = 0$

⇒ $P_y = Q_x$

(i.e) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Problem

Show that the equation z = px+qy is compatible with any equation f(x,y,z,p,q) = that is homogeneous in x,y and z. Solve completely that simultaneous equations z = px+qy, $2xy(p^2+q^2) = z(yp+xq)$.

If f is a homogeneous function in x,y,z of degree n, then by Euler's theorem, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf$].

Solution:

f(x,y,z,p,q) = 0, where f is a homo. function of x,y,z of degree n.

$$\therefore x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z} = nf$$

Here n = o

$$\therefore x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z} = 0$$

$$xfx+yfy+zfz = 0 \qquad \ldots\ldots (1)$$

The other equation g = px+qy-z

$g_x = p \qquad g_p = x$

$g_y = q \qquad g_p = y$

$g_z = -1$

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}$$

$= fxg_p-gxf_p+p(f_zg_p-f_pg_z)+(f_yg_q-f_qg_y)+q(f_zg_q-g_zf_q)$

$= xf_x-pf_p+p(f_zx-f_p(-1))+(yf_y-qf_q)+q(f_zy-f_q(-1))$

$= xf_x-pf_p+pf_zx+pf_p+yf_y-qf_q+f_zqy+qf_q$

$= xf_x+pxf_z+yf_y+f_zqy$

$= xf_x+yf_y+(px+qy)f_z$

$= xf_x+yf_y+zf_z$

$= 0$ [using (1)]

$\therefore$ [f, g] = 0

∴The equations are compatible.

Given

$$z = px+qy \qquad \ldots\ldots (1)$$

$$2xy(p^2+q^2) = z(yp+xq)$$

$$2xy(p^2+q^2) = (px+qy)(yq+xq)$$

$$2xy(p^2+q^2) = p^2xy+x^2pq+y^2pq+q^2xy$$

$$2xy(p^2+q^2) = xy(p^2+q^2)+pq(x^2+y^2)$$

$$2xy(p^2+q^2)-xy(p^2+q^2) = pq(x^2+y^2)$$

$$xy(p^2+q^2) = pq(x^2+y^2)$$

$$\frac{xy}{x^2+y^2} = \frac{pq}{p^2+q^2}$$

$$px = z-qy$$

$$p = \frac{z-qy}{x}$$

$$2xy\left(\left(\frac{z-qy}{x}\right)^2+q^2\right) = z\left(\frac{z-qy}{x}\right)+xq$$

$$2xy\left(\frac{z^2+q^2y^2-2zqy+q^2x^2}{x^2}\right) = z\left(\frac{yz-qy^2+x^2q}{x}\right)$$

$$2yz^2+2q^2y^3-4zqy^2+2q^2x^2y = yz^2-qzy^2+zx^2q$$

$$2yz^2-yz^2+q^2[2y^3+2x^2y]+q[-4zy^2+qzy^2]-zx^2q = 0$$

$$yz^2+q^2y[2y^2+2x^2]-3zy^2q-zx^2q = 0.$$

Derive the equation of the Characteristic strip

Proof

Let p(x,y,z) be a point on the curve c. Let (x+dx,y+dy,z+dz) lies on the tangent plane to the elementary cone at p, if

$$dz = pdx+qdy \qquad \ldots\ldots.. (1)$$

Where p,q, satisfies the relation.

$F(x,y,z,p,q) = 0$ …….. (2)

Diff. (1) w.r.to p we get,

$$0 = \frac{dp}{dp}dx + \frac{dq}{dp}dy$$

$$0 = dx + \frac{dq}{dp}dy$$

$$\frac{dq}{dp}dy = -dx$$

$$\frac{dq}{dp} = \frac{-dx}{dy} \quad \text{…….. (3)}$$

Diff. (2) w.r.to p

$$\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{dq}{dp} = 0$$

$$\Rightarrow \frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\left(\frac{-dx}{dy}\right) = 0 \quad \text{[using (3)]}$$

$$\Rightarrow F_p + F_q\left(\frac{-dx}{dy}\right) = 0$$

$$\Rightarrow F_p = F_q . \frac{dx}{dy}$$

$$\Rightarrow F_p\, dy = F_q\, dx$$

$$\Rightarrow F_p\, dy = F_q\, dx$$

$$\Rightarrow \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{pdx + qdy}{pF_p + pF_q}$$

$$\Rightarrow \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q}$$

This shows that $x'(t)$, $y'(t)$ and $z'(t)$ are proportional to F_p, F_q and pF_p+qF_q

Now, $$p'(t) = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial p}{\partial x}x'(t)+\frac{\partial p}{\partial y}y'(t)$$

$$p'(t) = \frac{\partial p}{\partial x}F_p+\frac{\partial p}{\partial y}F_q \quad \text{........ (4)}$$

Diff. (2) para. w.r.to x, we get,

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial F}{\partial p}\frac{\partial p}{dx}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}= 0$$

$$\Rightarrow F_x+F_z p+F_p\frac{\partial p}{\partial x}+F_q\frac{\partial q}{\partial x} = 0$$

$$\Rightarrow F_p\frac{\partial p}{\partial x}+F_q\frac{\partial p}{\partial x} = -(F_x+pF_z)$$

$$\Rightarrow F_p\frac{\partial p}{\partial x}+F_q\frac{\partial p}{\partial y} = -(F_x+pF_z) \qquad \left[\because \frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}\right]$$

$\therefore$ p'(t) = $-[F_x+pF_z]$ [by (4)]

|||rly We can prove, that

q'(t) = $-(Fy+qF_z)$

$\therefore$ The required equation for the determination of the characteristic strip are

$x'(t) = F_q$

$y'(t) = F_q$

$z'(t) = pF_p+qF_q$

$p'(t) = -[F_x+pF_z]$

$q'(t) = -[F_y+qF_z]$

These equations are known as characteristic equation of the diff equation F(x,y,z,p,q) = 0

Unit - V

Charpit's method

Charpit's method is the most general method of solving a P.D.E of the first order. Let $f(x,y,z,p,q) = 0$ ……(1) be the given equation.

If we know an equation of the form. $g(x,y,z,p,q) = 0$ ……(2). Which is compatible with (1), then solving (1) and (2) for p and q, we get,

$p = \varphi(x,y,a), \quad q = \psi(x,y,z)$

$dz = pdx+qdy$

We can get the soln of the given diff equ (1).

Charpit's method aims at getting an equation of the form (2) with a constant a.

(i.e) $g(x,y,z,p,q,q) = 0$ ……(3)

So that (1) and (3) are compatible

Since (1) and (3) are compatible. We get

$$[f, g] = 0$$

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

$$(f_x g_p - f_p g_x) + p(f_z g_p - f_p g_z) + (f_y g_q - f_q g_y) + q(f_z g_q - f_q - g_z) = 0$$

$$-f_p g_x - f_q g_y - (pf_p + qf_q)g_z + (f_x + pf_z)g_p + (f_y + qf_z)g_q = 0 \quad \text{……(4)}$$

For the determination of g

We know that the soln of (4) is same as the soln of Lagrange's auxillary equation.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} \quad \text{……(5)}$$

Solving the equation (5) we get p and q in the form $p = \varphi(x,,y,z,a)$, $q = \psi(x,y,z,a)$ use the value of p and q in

$$dz = pdx+qdy$$

Integrating we get soln of given equation as,

$$F(x,y,z,a,b) = 0$$

The solution involves two constants a and b, it is a complete solution of the given equation.

Note:

The equation (5) given above are known as Charpit's equation. We need not solve all the equation in (5).

We may choose those equation which convent give the values of p and q.

1. Find the complete integral of the equation $(p^2+q^2)y = qz$ by Charpit's method.

Solution:

Let $f = (p^2+q^2)y-qz$

$f_x = 0 \qquad f_p = 2py$

$f_y = p^2+q^2 \qquad f_q = 2qy-z$

$f_z = -q,$

We have, the Charpit's equation as

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + q(2qy - z)} = \frac{dp}{-[0 + p(-q)]} = \frac{dq}{-[p^2 + q^2 + q(-q)]}$$

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{-2p^2y + 2q^2y - qz} = \frac{dp}{pq} = \frac{dq}{-p^2 - q^2 + q^2}$$

$$\frac{dp}{pq} = \frac{dq}{-p^2}$$

$$p.dp = -q.dq$$

$$\int p\, dp = -\int q.\, dp$$

$$\Rightarrow \frac{p^2}{2} = -\frac{q^2}{2} + \frac{a}{2}$$

$$\Rightarrow \quad p^2+q^2 = a$$

Sub in the given equation

$$(p^2+q^2)y = qz$$

$$ay = qz$$

$$q = \frac{ay}{z}$$

$$p^2+q^2 = a$$

$$p^2 = q^2+a$$

$$p^2 = a - \frac{a^2y^2}{z^2}$$

$$p^2 = \frac{az^2 - a^2y^2}{z^2}$$

$$p = \frac{\sqrt{az^2a^2y^2}}{z^2}$$

$$dz = p\,dx + q.\,dy$$

$$= \frac{\sqrt{az^2a^2y^2}}{z^2}dx + \frac{ay}{z}dy$$

$$zdz = \sqrt{az^2 - a^2y^2}\,dx + ay.dy$$

$$z\,dz - ay\,dy = \sqrt{az^2 - a^2y^2}\,dx$$

$$\frac{z\,dz - ay\,dy}{\sqrt{az^2 - a^2y^2}} = dx$$

$$\frac{2a}{2a}\frac{z\,dz - ay\,dy}{\sqrt{az^2 - a^2y^2}} = dx$$

$$\frac{2azdz - 2a^2y\,dy}{2\sqrt{az^2 - a^2y^2}} = a.dx$$

$$\int \frac{2azdz - 2a^2 ydy}{2\sqrt{az^2 - a^2y^2}} = a.\int dx$$

(i.e) $\int d\sqrt{az^2 - a^2y^2} = a.\int dx$

$\Rightarrow \sqrt{az^2 - a^2y^2} = ax + b$

$\Rightarrow az^2 - a^2y^2 = (ax+b)^2$

$\Rightarrow az^2 = a^2y^2 + (ax+b)^2$

2. Find the complete integral of the equation $p^2x+q^2y = z$ by Charpit's method.

Solution:

$f = p^2x+q^2y-z$

$f_x = p^2$

$f_y = q^2$

$f_z = -1$

$f_p = 2px$

$f_q = 2py$

The auxillary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

(i.e) $\frac{dx}{2px} = \frac{dy}{2py} = \frac{dz}{p2px + q2qy} = \frac{dp}{-(p^2 + p(-1))} = \frac{dq}{-(q^2 + q(-1))}$

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2x + q^2y)} = \frac{dp}{p - p^2} = \frac{dq}{q - q^2}$$

$$\frac{p^2dx + 2pxdp}{p^2(2px) + 2px(p - p^2)} = \frac{q^2dy + 2qy.dq}{q^2(2qy) + 2qy(q - q^2)}$$

$$\frac{p^2dx + 2pxdp}{2p^3x + 2p^2x - 2p^3x} = \frac{q^2dy + 2qy.dq}{2q^3y + 2q^2y - 2q^3y}$$

$$\frac{p^2dx + 2pxdp}{2p^2x} = \frac{q^2dy + 2qy.dq}{2q^2y}$$

$$\int \frac{p^2dx + 2pxdp}{p^2x} = \int \frac{q^2dy + 2qy.dq}{q^2y}$$

$$\log (p^2x) = \log(q^2y) + \log a$$

$$\log (p^2x) = \log (q^2y)a$$

$$\Rightarrow \quad p^2x = q^2ya, \quad \text{where a is constant.}$$

Given equ is $p^2x+q^2y = z$

$$q^2ya + q^2y = z$$

$$q^2y(1+a) = z$$

$$q^2 = \frac{z}{y(1+a)}$$

$$q = \frac{1}{\sqrt{1+a}}\sqrt{\frac{z}{y}}$$

$$p^2x = q^2ya$$

$$= \frac{z}{y(1+a)}y$$

$$= \frac{az}{(1+a)}$$

$$p^2 = \frac{az}{x(1+a)}$$

$$\Rightarrow \quad p^2 = \sqrt{\frac{a}{a+1}}\sqrt{\frac{z}{x}}$$

Sub in the equation

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{\frac{a}{1+a}}\sqrt{\frac{z}{x}}\,dx + \frac{1}{\sqrt{1+a}}\sqrt{\frac{z}{y}}\,dy$$

$$\frac{dz}{\sqrt{z}} = \sqrt{\frac{a}{1+a}}(x)^{\frac{-1}{2}}dx + \frac{1}{\sqrt{1+a}}(y)^{\frac{-1}{2}}\,dy$$

$$\int (z)^{\frac{-1}{2}}dz = \sqrt{\frac{a}{1+a}}\int (x)^{\frac{-1}{2}}dx + \frac{1}{\sqrt{1+a}}\int (y)^{\frac{-1}{2}}\,dy$$

$$\frac{z^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} = \sqrt{\frac{a}{1+a}}\frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + \frac{1}{\sqrt{1+a}}\frac{y^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + b$$

$$\frac{z^{\frac{1}{2}}}{\frac{1}{2}} = \sqrt{\frac{a}{1+a}}\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{1}{\sqrt{1+a}}\frac{y^{\frac{1}{2}}}{\frac{1}{2}} + b$$

$$\sqrt{z} = \frac{\sqrt{ax}}{\sqrt{a+1}} + \frac{\sqrt{y}}{\sqrt{a+1}} + b$$

$$\sqrt{(a+1)}\sqrt{z} = \sqrt{ax} + \sqrt{y} + b$$

Which is the complete integral.

Special types of First order Equations.

Consider some special types of first-order para. diff. equation whose solutions may be obtained easily by Charpit's method.

Type I.

Equations involving only p and q

(i.e) The equations of the type f(p, q) = 0 ……(1)

Charpit's equations reduces to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

The solution of this equation is

p = a ……(2)

The corresponding value of q being obtained from (1) in the form

$$f(a, q) = 0 \qquad \ldots\ldots(1)$$

So that q = Q(a) a constant

∴ The solution of the equation is

$$z = ax+\varphi(a)y+b$$

Problem:

1. Find the complete integral of the equ pq = 1

Solution:

Given pq = 1

put p = a

$$\therefore \quad q = \frac{1}{p}$$

$$= \frac{1}{a}$$

$$q = \frac{1}{a}$$

∴ The complete soln is

$$z = ax+\frac{1}{a}y+b$$

$$z = \frac{a^2x+y+ab}{a}$$

$az = a^2x+y+ab$ Where a and be are constant.

2. Find the complete integral of the equ p+q = pq.

Solution:

Given p+q = pq

This is of the form f(p, q) = 0

put p = a

$\therefore$ a+q = aq

a = aq-q

q(a-1) = a

$$q = \frac{a}{a-1} = \varphi(a)$$

The complete soln is

$$z = ax + \frac{a}{a-1}y + b$$

Type II

Equation not involving the independent variables

(i.e) f(z,p,q) = 0 ……(1)

The Charpit's equation take the forms.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

$$\frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

$$\int \frac{dp}{p} = \int \frac{dq}{q}$$

log p = log q+log a

$\Rightarrow$ p = aq ……(2)

Solving (1) and (2) we get p and q.

Problem:

1. Find the complete integral of the equation $p^2z^2+q^2 = 1$

Solution:

Given $p^2z^2+q^2 = 1$ ……(1)

This is of the form f(z,p,q) = 0

$\therefore$ put p = qa

$\therefore (1) \Rightarrow q^2a^2z^2+q^2 = 1$

$q^2(1+a^2z^2) = 1$

$\Rightarrow q^2 = \frac{1}{1+a^2z^2}$

$\Rightarrow q = \frac{1}{\sqrt{1+a^2z^2}}$

$p^2z^2 + \frac{1}{1+a^2z^2} = 1$

$p^2z^2 = 1 - \frac{1}{1+a^2z^2}$

$= \frac{1+a^2z^2-1}{1+a^2z^2}$

$p^2z^2 = \frac{a^2z^2}{1+a^2z^2}$

$\therefore p^2 = \frac{a^2}{1+a^2z^2}$

$p = \frac{1}{\sqrt{1+a^2z^2}}$

We have

$dz = pdx+qdy$

$= \frac{a}{\sqrt{1+a^2z^2}}dx + \frac{1}{\sqrt{1+a^2z^2}}dy$

$\Rightarrow \sqrt{1+a^2z^2}\,dz = adx + dy$

2. Solve $z = p^2-q^2$

Solution:

Given $z = p^2-q^2$

This is of the form $f(z,p,q) = 0$

put $p = aq$

$$\therefore \quad z = a^2q^2-q^2$$

$$= (a^2-1)q^2$$

$$q^2 = \frac{z}{a^2-1}$$

$$q = \frac{\sqrt{z}}{\sqrt{a^2-1}}$$

$$p = aq$$

$$= \frac{a\sqrt{z}}{\sqrt{a^2-1}}$$

We have $dz = pdx+qdy$

$$dz = \frac{a\sqrt{z}}{\sqrt{a^2-1}}dx + \frac{\sqrt{z}}{\sqrt{a^2-1}}dy$$

$$\frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{a^2-1}}[adx+dy]$$

$$\int z^{\frac{-1}{2}}dz = \frac{1}{\sqrt{a^2-1}}\left\{\int adx + \int dy\right\}$$

$$\frac{z^{\frac{-1}{2}}}{\frac{1}{2}} = \frac{1}{\sqrt{a^2-1}}[ax+y+b]$$

$$2\sqrt{a^2-1}\sqrt{z} = (ax+y+b)$$

$$4z(a^2-1) = (ax+y+b)^2$$

3. Solve zpq = pq

Solution:

Given zpq = p+q

This is of the form f(z,p,q) = 0

put p = qa

$\therefore$ z.qz.q = qa+q

zaq^2 = qa+q

$\therefore zaq^2$ = q(a+1)

$$q = \frac{(a+1)}{za}$$

$$p = a.q$$

$$= a.\frac{(a+1)}{za}$$

$$\therefore p = \frac{a+1}{z}$$

We have dz = p.dx+q.dy

$$dz = \frac{a+1}{z}dx + \frac{a+1}{az}dy$$

$$z.dz = (a+1)\left[dx + \frac{1}{a}dy\right]$$

$\int$ ing

$$\frac{z^2}{2} = (a+1)\left[x + \frac{1}{ay}\right] + b$$

$$\therefore \quad z^2 = 2(a+1)\left(x + \frac{y}{a}\right) + b$$

4. Solve $z^2(1+p^2+q^2) = 1$

Solution:

This is of the form f(z,p,q) = 0

put p = qa

$$\therefore \quad z^2(1+q^2a^2+q^2) = 1$$

$$1+q^2a^2+q^2 = \frac{1}{z^2}$$

$$\therefore \quad q^2a^2+q^2 = \frac{1}{z^2}-1$$

$$q^2(1+a^2) = \frac{1-z^2}{z^2}$$

$$\therefore \quad q^2 = \frac{(1-z^2)}{z^2(1+a^2)}$$

$$q = \frac{\sqrt{(1-z^2)}}{z(1+a^2)}$$

$$p = aq$$

$$= \frac{a\sqrt{(1-z^2)}}{z(1+a^2)}$$

We have,

$$dz = pdx+q.dy$$

$$= \frac{a}{z}\frac{\sqrt{1-z^2}}{\sqrt{1+a^2}}dx+\frac{\sqrt{1-z^2}}{z\sqrt{1+a^2}}dy$$

$$\frac{z}{\sqrt{1-z^2}}dz = \frac{1}{\sqrt{1+a^2}}[adx+dy]$$

$$\frac{-2z}{-2\sqrt{1-z^2}}dz = \frac{1}{\sqrt{1+a^2}}[adx+dy]$$

$$-\left(\frac{-2z}{2\sqrt{1-z^2}}dz\right) = \frac{1}{\sqrt{1+a^2}}[adx+dy]$$

$$-d\left(\sqrt{1-z^2}\right) = \frac{1}{\sqrt{1+a^2}}[adx+dy]$$

$$-\int d\left(\sqrt{1-z^2}\right) = \frac{1}{\sqrt{1+a^2}}\left[\int adx+\int dy\right]$$

$$-\sqrt{1-z^2} = \frac{1}{\sqrt{1+a^2}}[ax+y+b]$$

$$-\sqrt{1-z^2}\sqrt{1+a^2} = [ax+y+b]$$

$$(1-z^2)(1+a^2) = (ax+y+b)^2$$

Type III

Separable Equations

A first order partial differential equation is said to be separable, if it can be written in the form

$$f(x,p) = g(y, q) \qquad \ldots\ldots(1)$$

∴ The Charpit's equation becomes

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{-g_y}$$

$$\frac{dx}{f_p} = \frac{dp}{-f_x}$$

$$\therefore \frac{dp}{dx} = \frac{f_x}{f_p} = 0$$

We have an ordinary diff. equ in x and p

Writing this equation in the form

$$f_p d_p + f_x d_x = 0$$

$$d[f(x,p)] = 0$$

$\therefore$ It is soln is $f(x, p) = a$

Hence we determine p, q from the relation

$$f(x, p) = a, \ g(y, q) = a$$

1. Find the complete integral of the equation $p^2y(1+x^2) = qx^2$

Solution:

Given $p^2y(1+x^2) = qx^2$

$$\Rightarrow \frac{p^2(1+x^2)}{x^2} = \frac{q}{y}$$

put $f(x, p) = g(y, q) = a$

$$\therefore \frac{p^2(1+x^2)}{x^2} = a^2 \qquad \frac{q}{y} = a^2$$

$$\Rightarrow \quad p^2 = \frac{a^2x^2}{1+x^2} \qquad \Rightarrow \quad q = ya^2$$

$$\Rightarrow \quad p = \frac{ax}{\sqrt{1+x^2}}$$

The soln is gn by the equation

$$dz = pdx+qdy$$

$$dz = \frac{ax}{\sqrt{1+x^2}}dx + a^2y.dy$$

$$dz = a.\frac{ax}{2\sqrt{1+x^2}}a^2y.dy$$

$$dz = a\left(d\sqrt{1+x^2}\right)+a^2y.dy$$

$\int$ ing

$$\int dz = a\int d\sqrt{1+x^2} + a^2\int y.dy$$

$$z = a\sqrt{1+x^2} + a^2\frac{y}{2} + b$$

2. Solve $p^2q(x^2+y^2) = p^2+q$

Solution:

$$\text{Given } p^2q(x^2+y^2) = p^2+q$$

$$p^2qx^2+p^2qy^2 = p^2+q$$

$$\Rightarrow x^2+y^2 = \frac{1}{q}+\frac{1}{p^2}$$

$$\text{(i.e)} \quad x^2-\frac{1}{p^2} = \frac{1}{q}-y^2$$

This is of the form $f(x, p) = g(y, q)$

$$x^2-\frac{1}{p^2} = a^2, \quad \frac{1}{q^2}-y^2 = a^2$$

$$\Rightarrow \quad x^2-a^2 = \frac{1}{p^2}, \quad \frac{1}{q^2} = y^2+a^2$$

$$p^2 = \frac{1}{x^2-a^2}, \quad q^2 = \frac{1}{y^2+a^2}$$

$$\therefore \quad p = \frac{1}{\sqrt{x^2-a^2}}, \quad q = \frac{1}{\sqrt{y^2+a^2}}$$

$$dz = pdx+qdy$$

$$dz = \frac{1}{\sqrt{x^2+a^2}}dx+\frac{1}{\sqrt{y^2+a^2}}.dy$$

$\int$ ing

$$z = \cosh^{-1}\left(\frac{x}{a}\right)+\frac{1}{a}\tan^{-1}\left(\frac{y}{a}\right)+b$$

3. $p^2q^2+x^2y^2 = x^2a^2(x^2+y^2)$

Solution:

Given $p^2q^2+x^2y^2 = x^2q^2(x^2+y^2)$

$\div$ by x^2q^2

$$\frac{p^2}{x^2}+\frac{y^2}{q^2} = x^2+y^2$$

$$\Rightarrow \quad \frac{p^2}{x^2}-x^2 = y^2-\frac{y^2}{q^2}$$

This is of the form $f(x, p) = g(y, q)$

$$\therefore \frac{p^2}{x^2}-x^2 = a^2, \quad y^2-\frac{y^2}{q^2} = a^2$$

$$\Rightarrow \frac{p^2}{x^2} = a^2+x^2, \quad \frac{y^2}{q^2} = y^2-a^2$$

$$\Rightarrow \quad p^2 = x^2(a^2+x^2), \quad q^2 = \frac{y^2}{y^2-a^2}$$

$$\Rightarrow \quad p = x\sqrt{a^2+x^2}, \quad q = \frac{y}{\sqrt{y^2-a^2}}$$

Consider the relation

$$dz = pdx+qdy$$

$$dz = x\sqrt{a^2+x^2}\,dx+\frac{y}{\sqrt{y^2-a^2}}.dy$$

$\int$ ing

$$\int dz = \frac{1}{2}\int 2x\sqrt{a^2+x^2}+\frac{1}{2}\int\frac{2y}{\sqrt{y^2-a^2}}dy$$

$$z = \frac{1}{2}\frac{(a^2+x^2)^{\frac{1}{2}}}{\frac{3}{2}}+\frac{1}{2}\frac{(y^2-a^2)^{\frac{-1}{2}}}{\frac{1}{2}}+b$$

$$\Rightarrow \quad z \quad = \quad \frac{\left(a^2+x^2\right)^{\frac{1}{2}}}{3}+\left(y^2-a^2\right)^{\frac{-1}{2}}+b$$

4. $p^2+q^2 = x^2+y^2$

Solution:

Given $p^2-x^2 = y^2-q^2$

This is of the form f(x, p) = g(y, q)

p^2-x^2	=	a^2,	y^2-q^2	=	a^2
p^2	=	a^2+x^2	q^2	=	y^2+a^2
∴ p	=	$\sqrt{a^2-x^2}$	∴ q	=	$\sqrt{y^2-a^2}$

dz = pd$x$+q.dy

$$dz = \sqrt{a^2+x^2}\,dx+\sqrt{y^2+a^2}\,dy$$

$$\int dz = \int\sqrt{a^2+x^2}\,dx+\int\sqrt{y^2+a^2}\,dy$$

$$= \frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\sin h^{-1}\left(\frac{x}{a}\right)+\frac{1}{2}y\sqrt{y^2+a^2}+\frac{1}{2}a^2\sinh^{-1}\left(\frac{y}{a}\right)+b$$

5. Solve px = qy

Solution:

Given px = qy

px	=	a,	qy	=	a
p	=	$\frac{a}{x}$	q	=	$\frac{a}{y}$

dz = pd$x$+qdy

$$dz = \frac{a}{x}dx+\frac{a}{y}dy$$

$$\int dz = a\int \frac{1}{x}dx + a\int \frac{1}{y}dy$$

`

z = a log$x$ + a logy + logb

z = a(log$x$ + logy)+b

z = a(logxy)+b

Type IV

Clairaut Equations

A given diff. equation of the form z = px+qy+f(p,q) is called the clariaut equation ……(1)

f_x = p $\quad$ f_p = x+f_p

f_y = q $\quad$ f_q = y+f_q

f_z = -1.

(i.e) F = px+qy+f(p,q)-z

∴ The corresponding Charpit's equations are

$$\therefore \frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p + qg_q} = \frac{dp}{-[f_x + pf_z]} = \frac{dq}{-[f_y + qf_z]}$$

$$\therefore \frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{p(x+f_p)+q(y+f_q)} = \frac{dp}{-\{p+p(-1)\}} = \frac{dq}{-[q+q(-1)]}$$

$$\frac{dp}{0} = \frac{dq}{0}$$

⇒ dp = 0 $\quad$ dq = 0

⇒ p = a $\quad$ ⇒ q = b

Where a and b are constants

Sub p = a, q = b in the Clairaut equation (1)

We get,

z = ax+by+f(a,b)

1. Find the complete integral of the equation (p+q) (z-px-qy) = 1

Solution:

Given (p+q) (z-px-qy) = 1

$$\Rightarrow z - px - qy = \frac{1}{p+q}$$

$$\therefore \quad z = px + qy + \frac{1}{p+q}$$

∴ The complete integral is

$$z = ax + by + \frac{1}{a+b}$$

2. Solve pqz = $p^2(xq+p^2)+q^2(yp+q^2)$

Solution:

Given pqz = $p^2(xq+p^2)+q^2(yp+q^2)$

÷ by pq

$$\Rightarrow \quad z = \frac{p}{q}(xq+p^2) + \frac{q}{p}(yp+q^2)$$

$$z = px + \frac{p^3}{q} + qy + \frac{q^3}{p}$$

$$\Rightarrow \quad z = px + qy + \frac{p^3}{q} + \frac{q^3}{p}$$

$$\text{(i.e)} \quad z = px + qy + \frac{p^4+q^4}{pq}$$

This is of z = px+qy+f(p,q) Clairaut's type

∴ The complete soln is

$$z = ax + by + \frac{a^4+b^4}{ab}$$

Solutions Satisfying Given Conditions

Consider the determination of surfaces which satisfy the partial differential equation

$$F(x,y,z,p,q) = 0 \quad \text{......... (1)}$$

and which satisfy some other condition such as passing through a given curve or circumscribing a given surface.

The solution of (1) which passes through a given curve c which has parametric equations,

$$x = x(t).\ y(t),\ z = z(t) \quad \text{......... (2)}$$

t being a parameter.

If there is an integral surface of the equation (1) through the curve c, then it is

a) A particular case of the complete integral

$$f(x,y,z,a,b) = 0 \quad \text{......... (3)}$$

obtained by giving a or be particular values.

(or)

b) A particular case of the general integral corresponding to (3) ie, the envelope of a one-parameter subsystem of (3) or.

c) The envelope of the two parameter system (3)

The points of intersection of the surface (3) and the curve c are determined in terms of the parameter t, by the equation.

$$f\{x(t),y(t),z(t),a,b\} = 0 \quad \text{......... (4)}$$

and the condition that the curve c should touch the surface (3) is that the equation (4) must have two equal roots or the equation (4) and the equation

$$\frac{\partial}{\partial t} f\{x(t), y(t), z(t), a, b\} = 0 \quad \text{......... (5)}$$

should have a common root.

The condition for this to be so is the eliminant of t from (4) and (5)

$$\psi(a,b) = 0 \quad \text{......... (6)}$$

Which is a relation between a and b alone

The equation (6) may be factoried into a set of equaions,

$b = \varphi_1(a), \qquad b = \varphi_2(a), \ldots.. \qquad \ldots\ldots\ldots (7)$

each of which defines a sub system of one parameter. The envelope of each of these one-parameter subsystem is a solution of the problem.

Jacobi's method

Solving the partial differential equation $F(x,y,z,p,q) = 0$ …..(1) depends on the fact that, if $u(x,y,z) = 0$ ….. (2) is a relation between x,y and z, then $p = \frac{-u_1}{u_3}$, ….. (3) $q = \frac{-u_2}{u_3}$, where ui denotes $\frac{\partial u}{\partial x}(i = 1,2,3)$.

If we substitute from equations (3) into the equation (1) we obtain a partial differential equation of the type

$f\{x,y,z,u_1,u_2,u_3\} = 0 \qquad \ldots\ldots\ldots (4)$

in which the new dependent variable u does not appear.

$$\frac{\partial f}{\partial u_1} = 2xu_1, \quad \frac{\partial f}{\partial u_2} = 2yu_2, \quad \frac{\partial f}{\partial u_3} = -2zu_3$$

$$\frac{\partial f}{\partial x} = u_1^{\,2}, \quad \frac{\partial f}{\partial y} = u_2^{\,2}, \quad \frac{\partial f}{\partial z} = u_3^{\,2}$$

The auxillary equations are,

$$\frac{dx}{fu_1} = \frac{dy}{fu_2} = \frac{dz}{fu_3} = \frac{du_1}{-fx} = \frac{du_2}{-fy} = \frac{du_3}{-fz}$$

$$\Rightarrow \frac{dx}{2u_1x} = \frac{dy}{2u_2y} = \frac{dz}{-2u_3z} = \frac{du_1}{-u_1^{\,2}} = \frac{du_2}{-u_2^{\,2}} = \frac{du_3}{-u_3^{\,2}}$$

Taking $\frac{dx}{2u_1x} = \frac{du_1}{-u_1^{\,2}}$

$$\Rightarrow \frac{dx}{2x} = \frac{du_1}{-u_1}$$

$\int$ ing

$$\int \frac{dx}{2x} = \int \frac{du_1}{-u_1}$$

$$\Rightarrow \log x = -2\log u_1 + \log a$$

$$\Rightarrow \log x + 2\log u_1 = \log a$$

$$\Rightarrow \log x + \log u_1^2 = \log a$$

$$\Rightarrow \log xu_1^2 = \log a$$

$$xu_1^2 = a$$

$$\therefore u_1 = \left(\frac{a}{x}\right)^{\frac{1}{2}}$$

Taking $\dfrac{dy}{2u_1 y} = \dfrac{du_2}{-u_2^2}$

$$\Rightarrow \frac{dy}{2y} = \frac{du_2}{-u_2}$$

$\int$ing

$$\int \frac{dy}{y} = -2\int \frac{du_2}{-u_2}$$

$$\log y + \log u_2^2 = \log b$$

$$\log yu_2^2 = \log b$$

$$\Rightarrow yu_2^2 = b$$

$$\therefore u_2 = \left(\frac{b}{y}\right)^{\frac{1}{2}}$$

The fundamental idea of Jacobi's is the introduction of two further partial differential equations of the first order.

$g(x,y,z,u_1,u_2,u_3,a) = 0$, $h(x,y,z,u_1,u_2,u_3,b) = 0$ ……… (5)

involving two arbitrary constants a and b such that,

a) Equations (4) and (5) can be solved for u_1,u_2,u_3

b) The equation $du = u_1dx+u_2dy+u_3dz$ ……… (6)

obtained from these values of u_1,u_2,u_3 is integrable.

The linear partial differential equation

$$fu_1\frac{\partial g}{\partial x}+fu_2\frac{\partial g}{\partial y}+fu_3\frac{\partial g}{\partial z}-fx\frac{\partial g}{\partial u_1}-fy\frac{\partial g}{\partial u_2}-fz\frac{\partial g}{\partial u_3}=0 \qquad \text{......... (7)}$$

Which has subsidiary equations,

$$\frac{\partial g}{fu_1}=\frac{\partial g}{fu_2}=\frac{\partial g}{fu_3}=\frac{\partial g}{-fx}=\frac{\partial g}{-fy}=\frac{\partial g}{-fz}=0 \qquad \text{......... (8)}$$

The procedure is the same as charpit's method.

Solve $p^2x+q^2y = z$, using Jocobi method.

Given $p^2x+q^2y = z$ .... (1)

$$p = \frac{-u_1}{u_3}, \qquad q = \frac{-u_2}{u_3}$$

$$p^2 = \frac{u_1^{\ 2}}{u_3^{\ 2}}, \qquad q^2 = \frac{u_2^{\ 2}}{u_3^{\ 2}}$$

$$(1) \Rightarrow \frac{u_1^{\ 2}}{u_3^{\ 2}}x+\frac{u_2^{\ 2}}{u_3^{\ 2}}y = z$$

$$\Rightarrow u_1^{\ 2}x+u_2^{\ 2}y-zu_3^{\ 2} = 0$$

$$zu_3^{\ 2} = xu_1^{\ 2}+yu_2^{\ 2}$$

$$u_3^{\ 2} = \frac{a+b}{2}$$

$$u_3 = \left(\frac{a+b}{2}\right)^{\frac{1}{2}}$$

$$du = u_1dx+u_2dy+u_3dz$$

$$du = \left(\frac{a}{x}\right)^{\frac{1}{2}}dx+\left(\frac{b}{y}\right)^{\frac{1}{2}}dy+\left(\frac{a+b}{z}\right)^{\frac{1}{2}}dz$$

$$\int du = \sqrt{a}\int\frac{1}{\sqrt{x}}\,dx+\sqrt{b}\int\frac{1}{\sqrt{y}}\,dy+\sqrt{a+b}\int\frac{1}{\sqrt{2}}\,dz$$

$$u = \sqrt{a}\ 2\sqrt{x} + \sqrt{b}\ 2\sqrt{y} + \sqrt{a+b}\ 2\sqrt{z} + c$$

$$u = 2\sqrt{ax} + 2\sqrt{by} + 2\sqrt{(a+b)z} + c$$

Partial Differential equations of the second order

1. The origin of second-order Equations

Suppose that the function z is given by an expression of the type

$$z = f(u)+g(v)+w \quad \ldots\ldots.. (1)$$

Where f and g are arbitrary functions of u and v respectively and u,v,w are the functions of *x* and y.

Then $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$ …….. (2)

Differential equations (1) parameter w.r.to. *x* and y.

$$\frac{\partial z}{\partial x} = f'(u)u_x + g'(v)v_x + w_x$$

$$\frac{\partial z}{\partial y} = f'(u)u_y + g'(v)v_y + w_y$$

(i.e) $p = f'(u)u_x + g'(v)v_x + w_x$

and $q = f'(u)u_y + g'(v)v_y + w_y$

Again Different these equations w.r.to. *x* and y

$$\frac{\partial^2 z}{\partial x^2} = f''(u)u_x^2 + f'(u)u_{xx} + g''(v)v_x^2 + g'(v)v_{xx} + w_{xx}$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''(u)u_x u_y + f'(u)u_{xy} + g''(v)v_x v_y + g'(v)v_{xy} + w_{xy}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(u)u_y^2 + f'(u)u_{yy} + g''(v)v_y^2 + g'(v)v_{yy} + w_{yy}$$

(i.e) $r = f''(u)u_x^2 + g'(v)v_x^2 + f'(u)u_{xx} + g'(v)v_{xx} + w_{xx}$

$$s = f''(u)u_x u_y + g''(v)v_x v_y + f'(u)u_{xy} + g'(v)v_{xy} + w_{xy}$$

$$r = f''(u){u_y}^2 + g'(v){v_y}^2 + f'(u)u_{yy} + g'(v)v_{yy} + w_{yy}$$

Now we have five equations involving the four arbitrary quantities f',f″,g',g″.

If we eliminate these four quantities from the five equations,

We obtain the relation.

$$\begin{vmatrix} p-w_x & u_x & v_x & 0 & 0 \\ q-w_y & u_y & v_y & 0 & 0 \\ r-w_{xx} & u_{xx} & v_{xx} & {u_x}^2 & {v_x}^2 \\ s-w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t-w_{yy} & u_{yy} & v_{yy} & {u_y}^2 & {v_y}^2 \end{vmatrix} = 0 \qquad \text{…….. (3)}$$

Which involves only the derivatives p,q,r,s,t and known functions of *x* and y.

∴ It is a partial differential equation of the second order.

If we expand the determinant on the L.H.S of equation (3) in terms of the elements of the first column, we obtain an equation of the form.

$$R_r+S_s+T_t+P_p+Q_q = W \qquad \text{…….. (4)}$$

∴ The relation (1) is a solution of the second - order linear partial differential equation (4).

Solve z = f(*x*+ay)+g(*x*-ay), where f and g are arbitrary functions and a is a constant.

Solution:

Given z = f(*x*+ay)+g(*x*-ay) …….. (1)

Differential (1) par. w.r.to. *x*

$$\frac{\partial z}{\partial x} = f'(x+ay) + g'(x-ay)$$

$$\frac{\partial z}{\partial y} = f'(x+ay)a + g'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay)a^2 + g''(x-ay)(-a)(-a)$$

$$= a^2\left[f''(x+ay) + g''(x-ay)\right]$$

(i.e) $$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

$$t = a^2 r$$

Similar methods apply in the case of higher - order equations. It is shown that any relation of the type.

$$z = \sum_{r=1}^{n} f_r(v_r)$$

Where the functions f_r are arbitrary and the functions v are known leads to a linear partial differential equations of the n^{th} order.

Linear partial differential Equations with constant coefficients

Consider the solution of linear partial differential equations with constant coefficients. An equation can be written in the form.

$$F(D,D') = f(x,y) \quad \ldots\ldots.. (1)$$

Where F(D,D') denotes the differential operator of the type.

$$F(D,D') = \sum_r \sum_s C_{rs} D^r D'^s \quad \ldots\ldots.. (2)$$

in which the quantities C_{rs} are constants and $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

The general solution corresponding to the homogeneous linear P.D.E

$$F(D,D')z = 0 \quad \ldots\ldots.. (3)$$

is called the complementary function of the equation (1)

|||rly any solution of the (1) is called a particular solution of (1).

Theorem:

If u is the complementary function and z_1, a particular integral of a linear partial differential equation then $u+z_1$ is a general solution of the equation

Proof

Consider the P.D.E.

$$F(D,D')z = f(x,y) \quad \ldots\ldots\ldots (1)$$

Let u is the complementary function of the given equation.

$$\therefore F(D,D')u = 0$$

Also, given z_1 is a particular integral of (1).

$$\therefore F(D,D')z = f(x,y)$$

∴ The general solution is

$$F(D,D')u+F(D,D')z_1 = 0+f(x,y)$$

(i.e) $F(D,D')(u+z_1) = f(x,y)$

$\Rightarrow u+z_1$ satisfies the equation (1)

$\Rightarrow u+z_1$ is the general solution of (1).

Theorem:

If $u_1,u_2,\ldots,u_n$ are solutions of the homogeneous linear P.D.E $F(D,D')z = 0$ then $\sum_{r=1}^{n} c_r u_r$ is also a solution where the c_r's are arbitrary constants.

Proof

The given homogeneous linear partial differential equation is

$$F(D,D')z = 0 \quad \ldots\ldots\ldots (1)$$

Given that $u_1,u_2,\ldots,u_n$ are the solution of (1).

$$\therefore \quad F(D,D')u_1 = 0$$

$$F(D,D')u_2 = 0$$

$$\vdots$$

$F(D,D')\,u_n \quad = \quad 0$

Also, $F(D,D')\,c_r u_r = c_r.F(D,D')v_r$. For any set of functions v_r.

Now,

$$F(D,D')\sum_{r=1}^{n} c_r u_r \quad = \quad \sum_{r=1}^{n} F(D,D')c_r u_r$$

$$= \quad \sum_{r=1}^{n} c_r\, F(D,D')u_r$$

$$= \quad c_1 F(D,D')\,u_1 + c_2\, F(D,D')u_2 + \dots + c_n\, F(D,D')u_n$$

$$= \quad 0$$

$\therefore \sum_{r=1}^{n} c_r\, u_r$ is the solution of (1)

Note:

The linear differential operator F(D,D') classify into two main types.

(a) F(D,D') is reducible if it can be written as the product of linear factors of the form D+aD'+b, where a and b are constants.

(b) F(D,D') is irreducible if it cannot be (written as above) decomposed into linear factors.

Theorem: 3

If the operator F(D,D') is reducible the order in which the linear factors occur is unimportant.

Proof

For proving this theorem, First we S.T

$$(\alpha_r D+\beta_r D'+\gamma_r)\,(\alpha_s D+\beta_s D'+\gamma_s) = (\alpha_s D+\beta_s D'+\gamma_s)\,(\alpha_r D+\beta_r D'+\gamma_r)$$

Now,

$$(\alpha_r D+\beta_r D'+\gamma_r)\,(\alpha_s D+\beta_s D'+\gamma_s) = \alpha_r\alpha_s D^2 + \alpha_r\beta_s DD' + \alpha_r\gamma_s D + \beta_r D'\alpha_s D + \beta_r\beta_r D'^2 + \beta_r\gamma_s D' + \gamma_r\alpha_s D + \gamma_r\beta_s D' + \gamma_r\gamma_s.$$

$$= \alpha_r\alpha_s D^2 + (\alpha_r\beta_s+\beta_r\alpha_s)DD' + \beta_r\beta_r D'^2 + (\alpha_r\gamma_s+\gamma_r\alpha_s)D + (\beta_r\gamma_s+\gamma_r\beta_s)D' + \gamma_r\gamma_s. \qquad \dots\dots\dots (1)$$

Also,

$(\alpha_s D+\beta_s D'+\gamma_s)(\alpha_r D+\beta_r D'+\gamma_r) = \alpha_r\alpha_r D^2 + (\alpha_r\beta_s+\alpha_r\gamma_s)DD'+\beta_r\beta_s D'^2 + (\alpha_r\gamma_s+\gamma_r\alpha_s)D+(\beta_r\gamma_s+\gamma_r\beta_s)D'+\gamma_r\gamma_s$ .... (2)

From (1) and (2) we get

$$(\alpha_r D+\beta_r D'+\gamma_r)(\alpha_s D+\beta_s D'+\gamma_s) = (\alpha_s D+\beta_s D'+\gamma_s)(\alpha_r D+\beta_r D'+\gamma_r)$$

∴ For any reducible operator can be written in the form.

$$F(D,D') = \prod_{r=1}^{n}(\alpha_r D+\beta_r D'+\gamma_r)$$

Theorem: 4

If $\alpha_r D+\beta_r D'+\gamma_r$ is a factor of F(D,D') and $\varphi_r(\xi)$ is an arbitrary function of the single variable ξ, then if $\alpha_r \neq 0$.

$u_r = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi_r(\beta_r x-\alpha_r y)$ is a solution of the equation F(D,D')z = 0.

Proof

We have,

$$u_r = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi_r(\beta_r x-\alpha_r y) \quad \text{........ (1)}$$

Differential equation (1) w.r.to *x*

$$Du_r = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi_r(\beta_r x-\alpha_r y)\beta_r + \varphi_r(\beta_r x-\alpha_r y)\left(\frac{-\gamma_r x}{\alpha_r}\right)\left(\frac{-\gamma_r}{\alpha_r}\right)$$

$$\therefore \quad Du_r = \beta_r \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x-\alpha_r y)\frac{-\gamma_r}{\alpha_r}u_r \quad \text{........ (2)}$$

Differential equation (1) w.r.to y

$$D'u_r = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x-\alpha_r y)(-\alpha_r)+\varphi_r(\beta_r x-\alpha_r y)0$$

$$\therefore \quad D'u_r = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x-\alpha_r y)(-\alpha_r) \quad \text{........ (3)}$$

(2)×α_r

$$\alpha_r Du_r = \alpha_r \beta_r \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y) - \gamma_r u_r \quad \text{........ (4)}$$

(3) $\times \beta_r$

$$\beta_r D' u_r = -\alpha_r \beta_r \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y) \quad \text{........ (5)}$$

(4) + (5) $\Rightarrow$

$$\alpha_r Du_r + \beta_r D' u_r = \alpha_r \beta_r \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y) - \gamma_r u_r - \alpha_r \beta_r \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi'(\beta_r x - \alpha_r y)$$

$\Rightarrow \alpha_r Du_r + \beta_r D'u_r = -\gamma_r u_r$

$\Rightarrow \alpha_r Du_r + \beta_r D'u_r + \gamma_r u_r = 0$

$\Rightarrow (\alpha_r D + \beta_r D' + \gamma_r) u_r = 0$ .... (6)

From the above theorem, we have

$$F(D, D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)$$

$$\therefore \quad F(D, D') = \left\{\prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)\right\}(\alpha_r D + \beta_r D' + \gamma_r) u_r \quad \text{........ (7)}$$

Combining equations (6) and (7)

$F(D,D')u_r = 0$

$\therefore$ u_r is a solution of $F(D,D')z = 0$

Theorem: 5

If $\beta_r D' + \gamma_r$ is a factor of F(D,D') and $\varphi_r(\xi)$, the if $\beta_r \neq 0$,

$u_r = \exp\left(\frac{-\gamma_r y}{\beta_r}\right)\varphi_r(\beta_r x)$ is a solution of the equation F(D,D') = 0.

Solution:

In the decomposition of F(D,D') into linear factors, we get multiple factors of the type $(\alpha_r D+\beta_r D'+\gamma_r)^n$

(i.e) T.P $(\alpha_r D+\beta_r D'+\gamma_r)^n z = 0$

If n = 2, then $(\alpha_r D+\beta_r D'+\gamma_r)^2 z = 0$ …….. (1)

Let $z = (\alpha_r D+\beta_r D'+\gamma_r)z$

then $(\alpha_r D+\beta_r D'+\gamma_r)\, z = 0$

By the above theorem, it has the solutions,

$$z = \exp\left(\frac{-\gamma_r x}{\alpha_r}\right)\varphi_r(\beta_r x-\alpha_r y)$$

If $\alpha_r \neq 0$

To find the corresponding function z, we have to solve the first order linear partial differential equations

$$\alpha_r \frac{\partial z}{\partial x}+\beta_r \frac{\partial z}{\partial y}+\gamma_r z = e^{\frac{-\gamma_r x}{\alpha_r}}\varphi_r(\beta_r x-\alpha_{ry})$$

This is of the form

$P_p+Q_q = R$

$P = \gamma_r$. $Q = \beta_r$, $R = -\gamma_r z + e^{\frac{-\gamma_r x}{\alpha_r}}\varphi_r(\beta_r x-\alpha_{ry})$

The auxillary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{\alpha_r} = \frac{dy}{\beta_r} = \frac{dz}{-\gamma_r z+e^{\frac{-\gamma_r}{\alpha_r}x}\varphi_r(\beta_r x-\alpha_r y)} \quad \text{…….. (2)}$$

Now, $\frac{dx}{\alpha_r} = \frac{dy}{\beta_r}$

$\Rightarrow \beta_r dx = \alpha_r dy$

$$\Rightarrow \quad \beta_r \int dx = \alpha_r \int dy$$

$$\Rightarrow \beta_r x = y + c_1$$

$$\Rightarrow \beta_r x - \alpha_r y = c_1$$

Sub this in the auxillary equation

$$\frac{dx}{\alpha_r} = \frac{dz}{-\gamma_r z + e^{\frac{-\gamma_r}{\alpha_r}x} \varphi_r(c_1)}$$

$$\frac{1}{\alpha_r}\left(-\gamma_r z + e^{\frac{-\gamma_r}{\alpha_r}x} \varphi_r(c_1)\right) = \frac{dz}{dx}$$

$$\alpha_r \frac{dz}{dx} = -\gamma_r z + e^{\frac{-\gamma_r}{\alpha_r}x} \varphi_r(c_1)$$

$$\frac{dz}{dx} = \frac{-\gamma_r}{\alpha_r} z + \frac{e^{\frac{-\gamma_r}{\alpha_r}x}}{\alpha_r} \varphi_r(c_1)$$

$$\frac{dz}{dx} + \frac{\gamma_r}{\alpha_r} z = \frac{1}{\alpha_r} e^{\frac{-\gamma_r}{\alpha_r}x} \varphi_r(c_1) \qquad \frac{dy}{dx} + P = Q$$

The solutions is $\qquad ye^{\int pdx} = \int Qe^{\int pdx} dx + c$

$$ze^{\int pdx} = \int Qe^{\int pdx} + c_2$$

$$e^{\int pdx} = e^{\int \frac{\gamma_r}{\alpha_r} dx} = e^{\frac{\gamma_r}{\alpha_r}\int dx} = e^{\frac{\gamma_r}{\alpha_r}x}$$

$$ze^{\frac{\gamma_r}{\alpha_r}x} = \int \frac{1}{\alpha_r} e^{\frac{-\gamma_r}{\alpha_r}x} \varphi_r(c_1) e^{\frac{\gamma_r}{\alpha_r}x} dx + c_2$$

$$ze^{\frac{\gamma_r}{\alpha_r}x} = \int \frac{\varphi_r(c_1)}{\alpha_r} dx + c_2$$

$$ze^{\frac{\gamma_r}{\alpha_r}x} = \frac{1}{\alpha_r}\int Q_r(c_r)dx + c_2$$

$$= \frac{1}{\alpha_r}\{\varphi_r(c_1)x + c_2\}$$

$$ze^{\frac{\gamma_r}{\alpha_r}x} e^{\frac{\gamma_r}{\alpha_r}x} = \frac{1}{\alpha_r}\{x\varphi_r(c_1) + c_2\}e^{\frac{-\gamma_r}{\alpha_r}x}$$

$$z = \frac{1}{\alpha_r}\{x\varphi_r(c_1) + c_2\}e^{\frac{-\gamma_r}{\alpha_r}x}$$

From (1) and (2) we get

$$z = x\varphi_r(\beta_r x - \alpha_r y) + \psi_r(\beta_{rx} - \alpha_r y)e^{\frac{-\gamma_r}{\alpha_r}x}$$

$$\text{Given } u_r = \exp\left(\frac{-\gamma_r y}{\beta_r}\right)\varphi_r(\beta_r x)$$

$$D'u_r = \varphi_r(\beta_r x)e^{\frac{-\gamma_r}{\beta_r}y - \frac{\gamma_r}{\beta_r}}$$

$$\beta_r D'u_r = -e^{\frac{-\gamma_r}{\beta_r}y}\varphi_r(\beta_r x)\gamma_r$$

$$= -u_r\gamma_r$$

$$\therefore \beta_r D'u_r + u_r\gamma_r = 0$$

$$(\beta_r D' + \gamma_r)\, u_r = 0$$

$$F(D,D') = \prod_{r=1}^{n}(\beta_r D' + \gamma_r)$$

$$F(D,D')u_r = \prod_{r=1}^{n}(\beta_s D' + \gamma_s)(\beta_r D' + \gamma_r)u_r$$

$$= 0$$

$\therefore$ u_r is the solutions of F(D,D') = 0

Theorem: 7

If $(\beta_r D'+\gamma_r)^m$ is a factor of F(D,D') and if the functions φ_{r1}, φ_{r2}, …., φ_{rm} are arbitrary, then $\exp\left(\frac{-\gamma_r y}{\beta_r}\right)\sum_{r=1}^{m} x^{s-1}\varphi_{rs}(\beta_r x)$ is a solution of F(D,D')z = 0.

$$F(D,D') = \prod_{r=1}^{n} (\alpha_r D + \beta_r D' + \gamma_r)^{mr}$$

The corresponding complementary any function is

$$u = \sum_{r=1}^{n} \exp\left(\frac{-\gamma_r}{\alpha_r} x\right) \sum_{s=1}^{mr} x^{s-1}\varphi_{rs}(\beta_r x - \alpha_r y)$$

Theorem: 6

If $(\alpha_r D+\beta_r D'+\gamma_r)^n$ $(\alpha_r \neq 0)$ is a factor of F(D,D') and if the functions φ_r, …. φ_{rn} are arbitrary, then,

$\exp\left(\frac{-\gamma_r}{\alpha_r} x\right)\sum_{s=1}^{n} x^{s-1}\varphi_{rs}(\beta_r x - \alpha_r y$ is a solution of F(D,D') and if the functions φ_{r1},….,φ_{rn} are arbitrary, then,

$\exp\left(\frac{-\gamma_r}{\alpha_r} x\right)\sum_{s=1}^{n} x^{s-1}\varphi_{rs}(\beta_r x - \alpha_r y$ is a solution of F(D,D') = 0.

Problem:

Solve the equation

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2 \partial y^2}$$

Solutions:

Given $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2 \partial y^2}$

$$\Rightarrow \frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} - 2\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0$$

$$\Rightarrow \quad \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)^2 = 0$$

This can be written as,

$$(D^2 - D^{12})z^2 = 0$$

$$[(D+D')(D-D')]^2 z = 0$$

$$\Rightarrow (D+D')^2 (D-D')^2 z = 0$$

$\therefore$ The solution is

$$z = x\varphi_1(x-y) + \varphi_2(x-y) + x\psi_1(x+y) + \psi_2(x+y)$$

Where the functions φ_1, φ_2, ψ_1, ψ_2 are arbitrary.

Find the solution of the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$

Solution:

Given $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$

This may be written as

$$(D^2 - D'^2)z = x - y$$

(i.e) $(D-D')(D+D')z = x - y$

The solution is $e^{\frac{-\gamma_r x}{\gamma_r}} \sum_{r=1}^{n} x^{r-1} \varphi_{rs}(\beta_r x - \alpha_r y$

$$(D-D')(D+D')z = 0$$

$\therefore$ The complementary functions is

$$e^0[\varphi_1(x+y) + \varphi_2(x-y)]$$

(i.e) $\varphi_1(x+y) + \varphi_2(x-y)$ …….. (1)

Where φ_1, φ_2 are arbitrary

To find the particular integral

$$(D-D')(D+D')z = x-y \quad \text{…….. (2)}$$

Take $z_1 = (D+D')z$ …….. (3)

$\therefore$ (2) $\Rightarrow (D-D')z_1 = x-y$

Which is the first order linear equation

$$\frac{\partial z_1}{\partial x} - \frac{\partial z_1}{\partial} = x-y$$

Which is of the from $P_p+Q_q = R$

The auxillary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{x-y}$$

Take $\frac{dx}{1} = \frac{dy}{-1}$

$$\int dx = -\int dy$$

$$\Rightarrow x = -y+c_1$$

$$\Rightarrow x+y = c_1 \quad \text{…….. (4)}$$

$$u = c_1$$

Also $\frac{dx-dy}{1-(-1)} = \frac{dz_1}{x-y}$

$$\frac{dx-dy}{1-(-1)} = \frac{dz_1}{x-y}$$

$$\frac{1}{2}(x-y)(dx-dy) = dz_1$$

$$\frac{1}{2}\int (x-y)(dx-dy) = \int dz_1$$

$$\frac{1}{2}\frac{(x-y)^2}{2} = z_1 + c_2$$

$$z_1 - \frac{1}{4}(x-y)^2 = c_2 \qquad \text{........ (5)}$$

$$v = c_2$$

Form (4) and (5)

$$f(u,v) = 0$$

$$\Rightarrow f(x+y), z_1 - \frac{1}{4}(x-y)^2 = 0$$

$$z_1 - \frac{1}{4}(x-y)^2 = f(x+y)$$

$$z_1 = \frac{1}{4}(x-y)^2 + f(x+y) \qquad \text{........ (6)}$$

Where f is arbitrary,

We may take f = 0,

$$\therefore \quad z_1 = \frac{1}{4}(x-y)^2$$

Sub the value of z_1 in equation (3)

$$(D+D')z = z_1$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)z = \frac{1}{4}(x-y)^2$$

$$\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = \frac{1}{4}(x-y)^2$$

This is of the form

$P_p + Q_q = R.$

$$P = 1,\ Q = 1,\ R = \frac{1}{4}(x-y)^2$$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\frac{1}{4}(x-y)^2}$$

Take

$$dx = dy$$

$$\int dx = \int dy$$

$$\therefore x = y + c_3$$

$$x - y = c_3 \qquad \text{........ (7)}$$

$$\frac{dx}{1} = \frac{dz}{\frac{1}{4}(x-y)^2}$$

$$dx = \frac{dz}{\frac{1}{4}c_3^{\,2}}$$

$$\int \frac{1}{4}c_3^{\,2}\,dx = \int dz$$

$$\frac{1}{4}c_3^{\,2}\int dx = \int dz$$

$$c_4 + \frac{1}{4}c_3^{\,2}x = z$$

$$z - \frac{1}{4}(x-y)^2 x = c_4 \qquad \text{........ (8)}$$

The solution is

$$f(u,v) = 0$$

$$f\left(x+y, z - \frac{1}{4}(x-y)^2 x\right) = 0$$

$$z - \frac{1}{4}(x-y)^2 x = f(x-y)$$

(i.e) $z = \frac{1}{4}(x-y)^2 x + f(x-y)$

Taking f = 0

The particular integral is

$\therefore \quad z = \frac{1}{4}x(x-y)^2$

Hence the general solution is

$$z = \frac{1}{4}x(x-y)^2 + \varphi_1(x+y) + \varphi_2(x-y)$$

Theorem: 8

$F(D,D')e^{ax+by} = F(a,b)e^{ax+by}$

Proof

We have,

$F(D,D') = C_{rs}D^rD'^s$

$\therefore \quad F(D,D')e^{ax+by} = CrsD^rD'^s(e^{ax+by})$ …….. (1)

$D^r(e^{ax+by}) = a^r(e^{ax+by})$

$D'^s(e^{ax+by}) = b^s(e^{ax+by})$

Now,

$\therefore C_{rs} D^rD's(e^{ax+by}) = C_{rs}\, a^rb^r(e^{ax+by})$

$\therefore \quad F(D,D')(e^{ax+by}) = F(a,b)\,(e^{ax+by})$

Using (1)

Hence the theorem.

Theorem: 9

$F(D,D')\,(e^{ax+by}\varphi(x,y)) = e^{ax+by}F(D+a,D'+b)\,\varphi(x,y)$

Proof

Find a particular integral of the equation.

$(D^2-D')z = 2y-x^2$

To find the P.I of $(D^2-D')z = 2y-x^2$

$$z = \frac{1}{D^2-D'}(2y-x^2)$$

$$= \frac{1}{-D'\left(1-\frac{D^2}{D'}\right)}(2y-x^2)$$

$$= \frac{-1}{D'}(2y-x^2)\left(1-\frac{D^2}{D'}\right)^{-1}$$

$$= \frac{-1}{D'}\left[1+\frac{D^2}{D'}+\frac{D^4}{D'^2}+....\right](2y-x^2)$$

$$= \frac{-1}{D'}(2y-x^2)\frac{-1}{D'^2}(-2)$$

$$= -\left[2\frac{y^2}{2}-x^2y-2\frac{y^2}{2}\right]$$

$$= -y^2+x^2y+y^2$$

$$= x^2y$$

Note:

When $f(x,y)$ is of the form e^{ax+by} . We obtain a particular integral is of the form $\frac{1}{F(a,b)}e^{ax+by}$ except if it happens that $F(a,b) = 0$.

Find a P.I of the equation $(D^2-D')z = e^{2x+y}$

Given $(D^2-D')z = e^{2x+y}$

In this case $F(D,D') = D^2-D'$

$a = 2$

$$b = 1$$

$$\therefore \quad F(a,b) = 3$$

$$\therefore \text{The P.I} \quad z = \frac{1}{3}e^{2x+y}$$

$$F(D,D') = D^2-D'$$

$$F(a,b) = 2^2-1$$

$$= 3$$

Find the particular integral of the equation $(D^2-D')z = A\cos(lx+my)$, where A,l,m are constants

Solution:

Given $(D^2-D')z = A\cos(lx+my)$

To find the particular integral

$$\text{Let} \quad z = c_1\cos(lx+my)+c_2\sin(lx+my)$$

Substitute in the given equation

$$(D^2-D')\, c_1\cos(lx+my)+c_2\sin(lx+my) = \left.\begin{aligned} &A\cos(lx+my)-c_1\cos(lx+my)l^2 \\ &-c_2\sin(lx+my)l^2+c_1\sin(lx+my)m \\ &-c_2\cos(lx+my)m \end{aligned}\right\} = A\cos(lx+my)$$

Equating the sine term to zero and the cosine term to A

$$-c_2l^2+c_1m = 0 \qquad \text{........ (1)}$$

$$-c_1l^2-c_2m = A \qquad \text{........ (2)}$$

To find c_1 and c_2 by solving (1) and (2)

$$(1) \times l^2 \Rightarrow \quad -c_2l^4+c_1ml^2 = 0$$

$$(2) \times m \Rightarrow \quad -c_2m^2-c_1ml^2 = Am$$

$$-c_2(m^2+l^4) = Am$$

$$\therefore c_2 = \frac{-Am}{m^2+l^4}$$

$$(1) \quad \Rightarrow \quad \frac{Aml^2}{m^2+l^4} + c_1 m = 0$$

$$mc_1 = \frac{-Aml^2}{m^2+l^4}$$

$$c_1 = \frac{-Al^2}{m^2+l^4}$$

$$z = c_1\cos(lx+my)+c_2\sin(lx+my)$$

$$= \frac{-Al^2}{m^2+l^4}\cos(lx+my) - \frac{Am}{m^2+l^4}\sin(lx+my)$$

$$= \frac{-A}{m^2+l^4}\left[l^2\cos(lx+my)\right] + m\sin(lx+my)$$

Equations with variable coefficient

Consider the equation of the type

$$Rr+Ss+Tt+f(x,y,z,p,q) = 0 \qquad \text{…….. (1)}$$

Which may be written in the form

$$L(z)+f(x,y,z,p,q) = 0 \qquad \text{…….. (2)}$$

Where L is the differential operator defined by the equation.

$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} \qquad \text{…….. (3)}$$

in which R,S,T, are continuous functions of x and y possessing continuous partial derivatives of higher order. By a suitable change of the independent variables we S.T any equation of the type (2) can be reduced to (1) of three canonical forms.

Suppose we change the independent variables from x,y to ξ, η where ξ = ξ(x,y) and η =η(x,y) and we write z(x,y) as ξ(ξ,η) then (1) takes the form.

$$A(\xi x, \xi y)\frac{\partial^2 \xi}{\partial \xi^2} + 2B(\xi x, \xi y, \eta x, \eta y)\frac{\partial^2 \xi}{\partial \xi \partial \eta} + A(\delta x, \delta y)\frac{\partial^2 \xi}{\partial \eta^2} = F(\xi, \eta, \zeta, \xi x, \xi y) \qquad \text{…….. (4)}$$

Where $A(u,v) = Ru^2+Suv+Tv^2$ …….. (5)

and $B(u_1,v_1,u_2,v_2 \quad = \quad Ru_1u_2+\frac{1}{2}S(u_1v_2,u_2,v_1)+Tv_1v_2.$ …….. (6)

The function F is derived from the given function f. The problem is to determine ξand η so that (4) takes the simplest form when the discriminant S^2-4RT of the quadratic form (5) is everywhere either positive, negative or zero we shall discuss these 3 cases.

Case (i)

$S^2-4RT > 0$

When this condition is satisfied the roots λ_1, λ_2 of the equations are

$$R\alpha^2+S\alpha+T \quad = \quad 0 \qquad \text{…….. (7)}$$

are real and distinct

And the coefficient of $\frac{\partial^2 s}{\partial \xi^2}$ and $\frac{\partial^2 s}{\partial \eta^2}$ in (4) will varnish.

If we choose ξ and ʒ such that,

$$\frac{\partial \xi}{\partial x}=\lambda_1\frac{\partial \xi}{\partial y}, \qquad \frac{\partial \eta}{\partial x}=\lambda_2\frac{\partial \eta}{\partial y}$$

Let us take $\xi = f_1(x,y)$, $\qquad \eta = f_2(x,y)$ …….. (8)

Where $f_1 = c_1$ and $f_2 = c_2$ are the solutions of the first order ordinary differential equation.

$$\frac{dy}{dx}+\lambda_1(x,y)=0, \qquad \frac{dy}{dx}+\lambda_2(x,y)=0, \qquad \text{…….. (9)}$$

In general

$$A(\xi x,\xi y)\, A(\eta x,\eta y)-B^2(\xi x,\xi y,\eta x,\eta y) \quad = \quad (4RT-S^2)\,(\xi x\eta y-\xi y\eta x) \qquad \text{…….. (10)}$$

When the A's are zero

$$B^2 \quad = \quad (S^2-4RT)\,(\xi x\eta y-\xi y\eta x)$$

Since $S^2-4RT > 0$

$\Rightarrow B^2 > 0$

Equation (1) is reduced to the form,

$$\frac{\partial^2 \zeta}{\partial \xi \eta} = \varphi(\xi, \eta, \zeta, \zeta x, \zeta y)$$

Reduce the equation $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

Given $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$

(i.e) r = x^{2t}

(i.e) r-x^{2t} = 0

R = 1, S = 0, T = $-x^2$

S^2-4RT > 0

0-4(1)($-x^2$) = $4x^2$ > 0.

To find the roots

$R\alpha^2+S\alpha+T = 0$

$\alpha^2+0-x^2 = 0$

$\alpha^2-x^2 = 0$

$\alpha^2 = x^2$

$\alpha = \pm x$

$\alpha_1 = x$, $\alpha_2 = -x$

$\frac{dy}{dx} + \lambda_1(x, y) = 0$, $\qquad \frac{dy}{dx} + \lambda_2(x, y) = 0$

$\Rightarrow \quad \frac{dy}{dx} + x = 0 \qquad \frac{dy}{dy} - x = 0$

$\Rightarrow \quad \frac{dy}{dx} = -x \qquad \frac{dy}{dx} = x$

dy = $-x$ dx $\qquad$ dy = x dx

$\int dy = -\int x\,dx \qquad \int dy = -\int x\,dx$

$y = \frac{-x^2}{2} + c_1$ $\qquad$ $y = \frac{x^2}{2} + c_2$

$y + \frac{x^2}{2} = c_1$ $\qquad$ $y - \frac{x^2}{2} = c_2$

(i.e) $\xi = y + \frac{x^2}{2}$, and $\eta = y - \frac{x^2}{2}$

$\frac{\partial \xi}{\partial x} = \frac{2x}{2}$ $\qquad$ $\frac{\partial \eta}{\partial x} = -\frac{2x}{2} = -x$

$\frac{\partial \xi}{\partial x} = x$ $\qquad$ $\frac{\partial \eta}{\partial y} = 1$

$\frac{\partial \xi}{\partial y} = 1$

A(u,v) = $Ru^2+Siv+Tv^2$

R = 1, S = 0, T = $-x^2$

∴ A (ξx,ξy) = $1.\xi x^2+0-x^2\xi y^2$

= $\xi x^2 - x^2$

= 0

$B(u_1,v_1;\ u_2,v_2)$ = $Ru_1u_2+\frac{1}{2}+S(u_1v_2+u_2v_1)+Tv_1v_2$

B(ξx,ξy;ηx,ηy)= $1\xi x\eta x+0-x^2\xi y\eta y$

= $x(-x)-x^2 1.1$

= $-x^2-x^2$

= $-2x^2$

A(ηx,ηy) = $\eta x^2-x^2\eta\eta$

= x^2-x^2

= 0

Sub in (4)

$$A(\xi x,\xi y)\frac{\partial^2\zeta}{\partial\xi^2}+2B(\xi x,\xi y,\eta x,\eta y)\frac{\partial^2\zeta}{\partial\xi\partial\eta}+A(\eta x,\eta y)\frac{\partial^2\zeta}{\partial\eta^2}=F(\xi,\eta,\zeta,\zeta x,\zeta y)$$

$$0+-2x^2(2)\frac{\partial^2\zeta}{\partial\xi\partial\eta}+0=0 \qquad \xi=y+\frac{x^2}{2},\eta=y-\frac{x^2}{2},\xi-\eta=\frac{2x^2}{2}$$

$$\Rightarrow -4x^2\frac{\partial^2\zeta}{\partial\xi\partial\eta}=0$$

$$\Rightarrow 4x^2\frac{\partial^2\zeta}{\partial\xi\partial\eta}=0$$

(i.e) $$4(\xi-\eta)\frac{\partial^2\zeta}{\partial\xi\partial\eta}=0 \qquad \text{........ (1)}$$

$z(x,y) = \zeta(\xi,\eta)$

$$\frac{\partial z}{\partial x} = \frac{\partial\zeta}{\partial\xi}.\frac{\partial\xi}{\partial x}+\frac{\partial\zeta}{\partial\eta}.\frac{\partial\eta}{\partial x}$$

$$= \frac{\partial\zeta}{\partial\xi}x+\frac{\partial\zeta}{\partial\eta}(1-x)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial\zeta}{\partial\xi}.1+x.\frac{\partial^2\zeta}{\partial\xi^2}\frac{\partial\xi}{\partial x}+\frac{\partial\zeta}{\partial\eta}(-1)+(-x)\frac{\partial^2 s}{\partial\eta^2}\frac{\partial\eta}{\partial x}$$

$$= \frac{\partial\zeta}{\partial\xi}+x\frac{\partial^2\zeta}{\partial\xi^2}x-\frac{\partial\zeta}{\partial\eta}-x\frac{\partial^2\zeta}{\partial\eta^2}(-x)$$

$$= \frac{\partial\zeta}{\partial\xi}+x^2\frac{\partial^2\zeta}{\partial\xi^2}-\frac{\partial\zeta}{\partial\eta}-x^2\frac{\partial^2\zeta}{\partial\eta^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial\zeta}{\partial\xi}.\frac{\partial\xi}{\partial y}+\frac{\partial\zeta}{\partial\eta}\frac{\partial\eta}{\partial y}$$

$$= \frac{\partial\zeta}{\partial\xi}.1+\frac{\partial\zeta}{\partial\eta}.1$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial\xi^2}+\frac{\partial^2\zeta}{\partial\eta^2}$$

$$\frac{\partial^2 z}{\partial x^2} - x^2\frac{\partial^2 z}{\partial y^2} = \frac{\partial \zeta}{\partial \xi} + x^2\frac{\partial^2 \zeta}{\partial \xi^2} - \frac{\partial \zeta}{\partial \eta} + x^2\frac{\partial^2 \zeta}{\partial \eta^2}$$

$$= \frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \eta}$$

$$\therefore \frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \eta} = 0 \qquad \ldots\ldots.. (2)$$

Combining (1) and (2)

$$4(\zeta - \eta)\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \xi}$$

$$\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \frac{1}{4(\xi - \eta)}\left(\frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \eta}\right)$$

Case (i)

S^2-4RT = 0

Here the roots of the equation (7) are equal

Pulting A(ξ*x*,ξy) and B = 0 and dividing by A(η*x*,ηy) the canonical form of (1) is.

$$\frac{\partial^2 \zeta}{\partial \eta^2} = \varphi(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta)$$

Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form.

Solution:

Given, $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

(i.e) r+2s+t = 0

R = 1, S = 2, T = 1

The equation is $R\alpha^2 + S\alpha + T = 0$

$$1.\alpha^2+2.\alpha+1 = 0$$

$$\alpha^2+2\alpha+1 = 0$$

$$(\alpha+1)^2 = 0$$

$$\alpha = -1, -1.$$

$$\frac{dy}{dx}+\lambda_1(x,y) = 0$$

$$\frac{dy}{dx}-1 = 0$$

$$\frac{dy}{dx} = 1$$

$$dy = dx$$

$$\int dy = \int dx$$

$$\Rightarrow x = y+c_1 \qquad ///\text{rly } x+y = c_2$$

$$\Rightarrow x-y = c_1$$

Let $\xi = x-y \qquad \eta = x+y$

$\xi x = 1 \qquad \eta x = 1$

$\xi y = -1 \qquad \eta y = 1.$

$A(u,v) = Ru^2+Suv+Tv^2$

$A(\xi x,\xi y) = B1.1+2(-1)+1.1$

$= 1-2+1$

$= 0$

$$B(u_1,v_1;\ u_2,v_2) = Ru_1u_2+\frac{1}{2}S(u_1v_2+u_2+v_1)+Tv_1v_2$$

$$B(\xi x,\xi y,\eta x,\eta y) = 1.\xi x\eta x+\frac{1}{2}.2(\xi x\eta y+\eta x\xi y)+1\xi y\eta y$$

$$= 1+1(1-1)+1.(-1)(1)$$

$$= \quad 1+0-1$$

$$= \quad 0.$$

$$A(\eta x,\eta y) = 1.\eta x^2+2\eta x\eta y+1.\eta y^2$$

$$= \quad 1.1^2+2.1.1+1.1$$

$$= \quad 1+2+1$$

$$= \quad 4.$$

$$(4) \Rightarrow 0+0+4\frac{\partial^2 \zeta}{\partial \eta^2} = F(\xi,\eta,\zeta,\zeta_\xi,\xi y)$$

$$4\frac{\partial^2 \zeta}{\partial \eta^2} = 0$$

$$\Rightarrow \frac{\partial^2 \zeta}{\partial \eta^2} = 0$$

Case (iii)

$$S^2-4RT < 0$$

The roots of equation (7) are complex. To get a real canonical form, we have the transformation

$$\alpha = \frac{1}{2}(\xi+\eta)$$

$$\beta = \frac{1}{2}i(\eta-\xi)$$

and it is shown that

$$\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \frac{1}{4}\left(\frac{\partial^2 \zeta}{\partial \alpha^2}+\frac{\partial^2 \zeta}{\partial \beta^2}\right)$$

∴ The canonical form is

$$\frac{\partial^2 \zeta}{\partial \alpha^2}+\frac{\partial^2 \zeta}{\partial \beta^2} = \psi(\alpha,\beta,\zeta,\zeta_\alpha,\zeta_\beta)$$

Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0 \text{ to a canonical form}$$

Solution:

Given $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$

$r + x^{2t} = 0$

$R = 1, \ S = 0, \ T = x^2$

The equation is $R\alpha^2 + S\alpha + T = 0$

$1.\alpha^2 + 0 + x^2 = 0$

$\alpha^2 + x^2 = 0$

$\alpha^2 = -x^2$

$\alpha = \pm ix$

$\lambda_1 = ix, \quad \lambda_2 = -ix$

We have

$\frac{dy}{dx} + \lambda_1(x, y) = 0$ $\qquad$ $\frac{dy}{dx} + \lambda_2(x, y) = 0$

$\frac{dy}{dx} = -\lambda_1$ $\qquad$ $\frac{dy}{dx} = -\lambda_2$

$\frac{dy}{dx} = -ix$ $\qquad$ $\frac{dy}{dx} = ix$

$dy = -ix\,dx$ $\qquad$ $dy = ix\,dx$

$\int dy = -i\int x dx$ $\qquad$ $\int dy = i\int x dx$

$y = -i\,x\frac{2}{2} + c_1$ $\qquad$ $y = ix\frac{2}{2} + c_2$

$y + i\,x\frac{2}{2} = c_1$ $\qquad$ $y - ix\frac{2}{2} = c_2$

xply by-i $\qquad$ xplybyi

$$-iy + i(-i)\frac{x^2}{2} = c_1 \qquad iy - (i)i\frac{x^2}{2} = c_2$$

$$-iy + i(-i)\frac{x^2}{2} = c_1 \qquad iy - (i)i\frac{x^2}{2} = c_2$$

Take $\xi = iy + \frac{x^2}{2}$ $\qquad \eta = -iy + \frac{x^2}{2}$

Given that,

$$\alpha = \frac{1}{2}(\xi + \eta) \qquad \beta = \frac{1}{2}i(\eta - \xi)$$

$\xi + \eta = x^2$ $\qquad \eta - \xi = - 2\ iy$

We have

$A(u, v) = Ru^2 + suv + Tv^2$

$A(\xi x, \xi y) = 1.x^2 + 0 + x^2\ i^2$

$= x^2 - x^2$

$= 0.$

$B(u_1, v_1; u_2, v_2) = Ru, u_2 + \frac{1}{2}$

$S(u_1 v_2 + u_2 v_1) + Tv_1 v_2$

$B(\xi x, \xi y; \eta x, \eta y) = 1.x.x + 0 + x^2(i^2)(-i)$

$= x^2 + x^2(-i^2)$

$= x^2 + x^2$

$= 2x^2$

$A(\eta x, \eta y) = 1.\eta x^2 + x^2 \eta y^2$

$= x^2 + x^2(-i)^2$

$= x^2 - x^2$

$= 0.$

Sub. in (4)

$$0+0+2.2x^2.\frac{\partial^2 s}{\partial\xi\partial\eta}=0$$

$$4x^2.\frac{\partial^2 s}{\partial\xi\partial\eta}=0$$

ie) $4(\xi+\eta)\frac{\partial^2 s}{\partial\xi\partial\eta}=0$ …..(1)

z(x, y) = S (α, β)

$$\frac{\partial z}{\partial x}=\frac{\partial s}{\partial\alpha}.\frac{\partial\alpha}{\partial x}+\frac{\partial s}{\partial\beta}.\frac{\partial\beta}{\partial x}$$

$1\alpha=\frac{1}{2}x^2, \beta=y$

$\alpha x = x$, $\beta x = 0$

$\alpha y = 0$, $\beta y = 1$

$$=\frac{\partial s}{\partial\alpha}.x.+\frac{\partial s}{\partial\beta}.0$$

$$=x.\frac{\partial s}{\partial\alpha}$$

$$\frac{\partial^2 z}{\partial x^2}=x.\frac{\partial^2 s}{\partial\alpha^2}\frac{\partial\alpha}{\partial x}+\frac{\partial s}{\partial\alpha}.1$$

$$=x^2.\frac{\partial^2 s}{\partial\alpha^2}+\frac{\partial s}{\partial\alpha}$$

$$\frac{\partial z}{\partial y}=\frac{\partial s}{\partial\alpha}.\frac{\partial\alpha}{\partial y}+\frac{\partial s}{\partial\beta}.\frac{\partial\beta}{\partial y}$$

$$=\frac{\partial s}{\partial\alpha}.0+\frac{\partial s}{\partial\beta}.1$$

$$=\frac{\partial s}{\partial\beta}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 s}{\partial\beta^2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + x^2.\frac{\partial^2 z}{\partial y^2} = x^2.\frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial s}{\partial \alpha} + x^2\frac{\partial^2 s}{\partial \beta^2}$$

$$0 = x^2\left(\frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial^2 s}{\partial \beta^2}\right) + \frac{\partial s}{\partial \alpha}$$

$$\therefore x^2\left(\frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial^2 s}{\partial \beta^2}\right) = -\frac{\partial s}{\partial \alpha}$$

$$\rightarrow \frac{\partial^2 s}{\partial \alpha^2} + \frac{\partial^2 s}{\partial \beta^2} = -\frac{1}{x^2}.\frac{\partial s}{\partial \alpha}$$

$$= -\frac{1}{2\alpha}.\frac{\partial s}{\partial \alpha}$$

$$[x^2 = 2\alpha]$$

Show how to find a solution containing two arbitrary functions of the equation s = f (x,y). Hence solve the equation s = 4xy + 1.

Solution:

Given s = f (x, y)

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f(x, y)$$

$$\int ing\ .\ y$$

$$\frac{\partial z}{\partial y} = \int_0^y f(x, y).dx + f_1(x)$$

Again integrating

$$z = \int_0^x \int_0^y f(x, y)\, dx\, dy + \int_0^x f_1(x)\, dy + f_2(g)$$

$$z=\int_0^x d\,\xi\int_0^y f(\xi,\eta)\,d\eta)+[\,f_1(x)y]_0^x+f_2(y)$$

$$=\int_0^x d\,\xi\int_0^y f(\xi,\eta)\,d\eta+x\,f_1(x)+f_2(y)$$

$$=\int_0^x d\,\xi\int_0^y f(\xi,\eta)\,d\eta+\alpha\;fn\,qx+f_2(y)$$

$$=\int_0^x d\,\xi\int_0^y f(\xi,\eta)\,d\eta+f_1(x)+f_2(y)$$

Given s = 4xy + 1

f(ξ, η) = 4 ξη + 1

$$z=\int_0^x d\,\xi\int_0^y f(\xi,\eta)\,d\eta+f_1(x)+f_2(y)$$

$$=\int_0^x d\,\xi\int_0^y (4\xi\eta+1)\,d\eta+f_1(x)+f_2(y)$$

$$=\int_0^x d\,\xi\left[4^2\,\xi\frac{\eta^2}{2}+\eta\right]_0^y+f_1(x)+f_2(y)$$

$$=\int_0^x d\,\xi[2\xi\,y^2+y]+f_1(x)+f_2(y)$$

$$=\int_0^x 2\xi\,y^2\,d\xi+\int_0^x y\,d\,\xi+f_1(x)+f_2(y)$$

$$=2y^2\left[\frac{\xi^2}{2}\right]_0^x+y[\xi]_0^x+f_1(x)+f_2(y)$$

$= x^2\, y^2 + xy + f_1(x) = f_2(y)$

$= xy\,(xy+1) + f_1(x) + f_2(y)$.

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